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C_p-PERTURBATIONS OF NEST ALGEBRAS

GARETH J. KNOWLES

ABSTRACT. Given the ideal $\mathcal{C}_p$ and a nest algebra $\mathcal{A}$ in $\mathcal{L}(H)$ there are two corresponding subalgebras of $\mathcal{L}(H)$. The first consists of all $\mathcal{C}_p$-perturbations of $\mathcal{A}$. The second, a natural generalization of the quasitriangular algebra corresponding to $\mathcal{A}$, consists of all T in $\mathcal{L}(H)$ with $E \to (I-E)TE$ continuous from Lat $\mathcal{A}$ into $\mathcal{C}_p$. Necessary and sufficient conditions are given for these algebras to be identical.

1. Introduction. The notion of a quasitriangular algebra corresponding to some arbitrary nest of projections was first introduced in [**3**]. In [**3**], a particularly useful characterization of a quasitriangular algebra as the collection of all compact perturbations of the corresponding nest algebra, is given. From this characterization many useful developments followed [**1, 3**]. This paper is concerned with studying a class of algebras, the quasitriangular p-classes, that are a natural generalization of a quasitriangular algebra. The main result contained here is that the Arveson characterization fails for $1 \leq p < \infty$. Necessary and sufficient conditions are given for the characterization to hold. This research was undertaken while in receipt of a Science Research Council award during the preparation of the authors Ph.D. under the supervision of J. Erdos.

A totally ordered collection of subspaces of some Hilbert space H closed under the taking of suprema and infima is termed a (complete) nest. Given such a nest $\mathcal{N}$ of subspaces there is a corresponding totally ordered set of projections on the subspaces on $\mathcal{N}$ which will be denoted by $\mathcal{E}$. Following [**3**], the quasitriangular algebra corresponding to such a nest $\mathcal{N}$ consists of all $T \in \mathcal{L}(H)$ (the collection of all bounded linear operators acting on H) with the property that $E \to (I-E)TE$ is a continuous map from $\mathcal{E}$, with the strong operator topology, to $\mathcal{C}_\infty$, the ideal of all compact operators with the norm topology. The algebra Alg $\mathcal{N}$ of all operators acting on H leaving each subspace in $\mathcal{N}$ invariant will be denoted by $\mathcal{A}$.

DEFINITION. An operator T in $\mathcal{L}(H)$ is said to be in the quasitriangular p-class corresponding to $\mathcal{N}$, denoted by $\mathcal{T}_p$ $(= \mathcal{T}_p(\mathcal{N}))$ whenever

(i) $(I-E)TE$ is in $\mathcal{C}_p$ for each E in $\mathcal{E}$ and
(ii) $E \to (I-E)TE$ is continuous from $\mathcal{E}$, with the strong operator topology to $\mathcal{C}_p$, with the p-norm.

The proof of the following is based on that of [**3**] for $\mathcal{C}_\infty$.

LEMMA 1. *The algebra $\mathcal{A} + \mathcal{C}_p$ is a subalgebra of $\mathcal{T}_p$.*

PROOF. Certainly if $T = A + K$ for some $A \in \mathcal{A}$ and $K \in \mathcal{C}_p$ then $(I-E)TE$ is in $\mathcal{C}_p$. Suppose that $0 \neq K = x \otimes y$ for some x, y in H. Then

$$(I-E)KE - (I-F)KF = -Ex \otimes (E-F)y + (E-F)x \otimes (I-F)y.$$

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Thus

$$\|(I-E)KE-(I-F)KF\|_p \le \|(E-F)y\|\cdot\|x\| + \|(E-F)x\|\cdot\|y\|$$

from which (ii) now follows: Let R be a finite rank operator. Then by linearity it follows that (ii) holds for $K=R$. Choose R_n finite rank with $\|K-R_n\|_p \to 0$. For each $E\in\mathcal{E}$,

$$\|(I-E)KE\|_p \le \|(I-E)(K-R_n)E\|_p + \|(I-E)R_nE\|_p.$$

Since $\|(I-E)(K-R_n)E\|_p \to 0$ as $n\to\infty$ it follows that the map $E\to(I-E)KE$ is the uniform limit of continuous functions, and hence continuous.

COROLLARY 2. *When $\mathcal{N}$ is a finite nest, then $\mathcal{A}+\mathcal{C}_p=\mathcal{T}_p$.*

PROOF. By Lemma 1 it will be sufficient to show $\mathcal{A}+\mathcal{C}_p\supseteq\mathcal{T}_p$. Let $\mathcal{N}$ be the nest $0=E_0<E_1<\cdots<E_m=I$. For any $T\in\mathcal{T}_p$

$$T=\sum_{n=1}^{m}(I-E_n)T(E_n-E_{n-1})+\sum_{n=1}^{m}(E_n-E_{n-1})T(I-E_{n-1}),$$

where the first summand is in $\mathcal{C}_p$ and the second in $\mathcal{A}$.

THEOREM 3. *If $\mathcal{N}$ is an infinite nest then $\mathcal{A}+\mathcal{C}_p$ is strictly contained in $\mathcal{T}_p$ for $1\le p<\infty$.*

PROOF. Using the well-known fact that $\mathcal{E}$ is compact in the strong operator topology, a sequence of distinct projections can be found with either $E_n s\nearrow P$ or $E_n\searrow sP$ for some P in $\mathcal{E}$. Let us assume that $E_n s\nearrow P$. For each $n\in N$, choose x_n a unit vector in the range of (E_n-E_{n-1}) and define the bounded operator K in $\mathcal{L}(H)$ by

$$K=\sum_{i=1}^{\infty} i^{-1/2}x_i\otimes x_{i+1}.$$

The following will now be shown to hold true for this K:

(i) $(I-E)KE$ is in $\mathcal{C}_p$ for each E in $\mathcal{E}$; $1\le p\le\infty$.

(ii) $E\to(I-E)KE$ is continuous from $\mathcal{E}$ with the strong operator topology to $\mathcal{C}_1$ with the trace-class norm.

(iii) $K\notin\mathcal{A}+\mathcal{C}_2$.

(i) Fix $E\in\mathcal{E}$ with $E<P$; then by assumption we can find an n with $E<E_n<P$, giving $(I-E)KE=\sum_{i=1}^{n} i^{-1/2}Ex_i\otimes(I-E)x_{i+1}$ is in $\mathcal{C}_p$ for $1\le p\le\infty$.

(ii) Let $E\in\mathcal{E}$ satisfy $E<P$. It follows that, for some $n\in N$, $E_{n-1}<E\le E_n$. Given $\varepsilon>0$ choose F_1 as follows: when E has an immediate predecessor E_- let F_1 be E_-; when E has no immediate predecessor choose F_1 in $\mathcal{E}$ with $E_{n-1}<F_1<E$ and $\|(E-F_1)x_j\|<\varepsilon/8$ for $j=1,2,\ldots,n+1$. For an F with $F_1<F\le E$ it now follows that

$$\begin{aligned}(I-E)KE&-(I-F)KF\\ &=(n-1)^{-1/2}(Ex_{n-1}\otimes(I-E)x_n-Fx_{n-1}\otimes(I-F)x_n)\\ &\quad+n^{-1/2}(Ex_n\otimes(I-E)x_{n+1}-Fx_n\otimes(I-F)x_{n+1})\\ &=(n-1)^{-1/2}[-Ex_{n-1}\otimes(E-F)x_n+(E-F)x_{n-1}\otimes(I-F)x_n]\\ &\quad+n^{-1/2}[-Ex_n\otimes(E-F)x_{n+1}+(E-F)x_n\otimes(I-F)x_n].\end{aligned}$$

Let T be the first term in the above expansion, that is

$$T = (n-1)^{-1/2}(-Ex_{n-1} \otimes (E-F)x_n).$$

Thus TT^* is a rank one operator with $\sigma[(TT^*)^{1/2}] = (n-1)^{-1/2}\|Ex_{n-1}\| \cdot \|(E-F)x_n\|$. From this it now follows that $\|T\|_1 < \varepsilon/8$. Similar computations for the other three terms will now show that $\|(I-E)KE - (I-F)KF\|_1 < \varepsilon/2$ for $F_1 < F \leq E$.

Similarly, when $E_k \leq E < E_{k+1}$ for some k, given $\varepsilon > 0$, choose F_2 as follows: when E has an immediate successor E_+ let F_2 be E_+; when E has no immediate successor choose F_2 in $\mathcal{E}$ with $E < F_2 < E_{k+1}$ satisfying

$$\|(F_2 - E)x_j\| < \varepsilon/8 \quad \text{for } j = 1, 2, \ldots, k+1.$$

An exactly similar argument to the one above will show that whenever G in $\mathcal{E}$ satisfies $E \leq G < F_2$ then $\|(I-G)KG-(I-E)KE\|_1 < \varepsilon/2$. Thus, it is concluded that $\|(I-G)KG - (I-F)KF\|_1 < \varepsilon$ for $F, G \in (F_1, F_2)$, showing continuity at $E < P$. Note that, for $E \geq E_n$, $\|(I-E)KE\|_1 \leq 2n^{-1/2}$ which implies continuity at P. A parallel argument deals with the case that there is a sequence of distinct projections $E_n \searrow sP$ in $\mathcal{E}$.

(iii) First note that $\|K\|_2^2 = \sum_{j=1}^{\infty} \|(E_{j+1} - E_j)K(E_j - E_{j-1})\|_2^2 = \sum_{j=1}^{\infty} j^{-1}$, showing K is not in C_2. Suppose $\mathcal{F}$ is some arbitrary nest of projections acting on some Hilbert space H and X acting on H is in $\mathcal{C}_p$ $(1 < p < \infty)$. It is shown in [**2**, Theorem 3.2] that if X is rewritten as $X = \mathcal{L}(X) + \mathcal{D}(X) + \mathfrak{A}^{\perp}(X)$ where

$$[\mathcal{L}(X^*)]^* = \mathfrak{A}^{\perp}(X) = \lim_{\mathcal{P}} \left\{ \sum_{P_n \in \mathcal{P}} (P_n - P_{n-1})XP_{n-1} \right\},$$

$$\mathcal{D}(X) = \lim_{\mathcal{P}} \left\{ \sum_{P_n \in \mathcal{P}} (P_n - P_{n-1})X(P_n - P_{n-1}) \right\},$$

$$\mathcal{L}(X) = \lim_{\mathcal{P}} \left\{ \sum_{P_n \in \mathcal{P}} P_{n-1}X(P_n - P_{n-1}) \right\}$$

(the limit being taken over all finite partitions $\mathcal{P}$ of $\mathcal{F}$), then these define bounded operators, in fact they are also in $\mathcal{C}_p$. Applying this to the above, it is first noted that $K = \mathfrak{A}^{\perp}(K)$ and $\mathfrak{A}^{\perp}(A) = 0$ for any A where we decompose K relative to $\mathcal{E}$. Suppose then that it were possible to write K in the form $A + C$ for some $A \in \mathcal{A}$ and $C \in \mathcal{C}_2$. Therefore,

$$K = \mathfrak{A}^{\perp}(K) = \mathfrak{A}^{\perp}(K - A) = \mathfrak{A}^{\perp}(C)$$

which is in $\mathcal{C}_2$, contradicting the above. This completes the argument in the case $p = 2$. It is easily seen that a simple adjustment of the eigenvalues in the definition of K will provide an analogous counterexample for a general p, $1 < p < \infty$.

The case for $p = 1$ now follows from that of $p = 2$. Suppose K, as above, were of the form $K = A + C$ for some A in $\operatorname{Alg}\mathcal{E}$ and C in $\mathcal{C}_1$. Decompose C in $\mathcal{C}_2$ as before. It will now follow that K is in C_2, a contradiction.

DEFINITION. For each $T \in \mathcal{A} + \mathcal{C}_p$ $(\infty > p > 1)$ set

$$|||T|||_p = \|A + \mathcal{L}(C) + \mathcal{D}(C)\| + \sup_{E \in \mathcal{E}} \|(I-E)TE\|_p$$

where $T = A + C$ and C decomposes into $\mathcal{L}(C) + \mathcal{D}(C) + \mathfrak{A}^{\perp}(C)$ with respect to $\mathcal{E}$.

LEMMA 4. *$|||\cdot|||_p$ is a norm on $\mathcal{A}+\mathcal{C}_p$.*

PROOF. It is obviously well defined, subadditive and satisfies $|||\alpha T|||_p = |\alpha|\cdot|||T|||_p$. Suppose then that $|||T|||_p = 0$. It follows from [**1**, Theorem 1.1] that T is in $\mathcal{A}$. If $T = A + C$ for some $A \in \mathcal{A}$, $C \in \mathcal{C}_p$

$$T = A + C = A + \mathcal{L}(C) + \mathcal{D}(C),$$

from which it follows that $T = 0$.

REMARKS. It is not very hard to show that $\mathcal{A}+\mathcal{C}_p$ is not complete in the $|||\cdot|||_p$ norm. For example, take a sequence $\{A_n\}_{n=1}^{\infty}$ where

$$A_n = \sum_{j=1}^{n} b_j(x_j \otimes x_{j+1}) \quad \text{for some } \{b_n\} \in c_0 \setminus l_p.$$

It would be of interest to know whether its completion in this norm is $\mathcal{T}_p$.

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DEPARTMENT OF MATHEMATICS, TEXAS TECH UNIVERSITY, LUBBOCK, TEXAS 79409