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NECESSARY CONDITIONS FOR STABILITY OF NONSINGULAR ENDOMORPHISMS OF THE CIRCLE

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ABSTRACT. In this article we prove that Axiom A is a necessary condition for structural stability of C^1 nonsingular endomorphisms of the circle.

1. Introduction. We say that an endomorphism $f\colon S^1 \to S^1$ of the circle is *nonsingular* if df is injective at each point of S^1. Let $N(S^1)$ be the space of C^1 nonsingular endomorphisms of S^1 endowed with the C^1 topology. f in $N(S^1)$ is said to be *structurally stable* if it has a neighborhood $\mathcal{U}$ such that any $g \in \mathcal{U}$ is topologically conjugate to f; i.e., there exists a homeomorphism $h\colon S^1 \to S^1$ satisfying $hf = gh$. In [**4**] Z. Nitecki proves that Axiom A is a sufficient condition for a nonsingular endomorphism of S^1 to be structurally stable. Recall that $f \in N(S^1)$ satisfies Axiom A if

(a) $\overline{\operatorname{Per}(f)} = \Omega(f)$. Here $\operatorname{Per}(f)$ denotes the set of all periodic points of f and $\Omega(f)$ the set of nonwandering points of f, i.e., $x \in \Omega(f)$ if and only if for any neighborhood U of x there is an integer $n > 0$ such that $f^n(U) \cap U \neq \emptyset$.

(b) $\Omega(f)$ has a hyperbolic structure, i.e., $\Omega(f)$ decomposes into a disjoint union of two closed, invariant subsets $\Omega(f) = \Omega_c \cup \Omega_e$ such that there exist $k > 0$, $0 < \lambda < 1$, satisfying

$$|df^n(x)| \leq k\lambda^n \quad \text{for all } x \in \Omega_c,$$

and

$$|df^n(x)| \geq k\lambda^{-n} \quad \text{for all } x \in \Omega_e.$$

The purpose of this paper is to prove that the condition above is necessary for structural stability.

THEOREM. *If $f \in N(S^1)$ is structurally stable then f satisfies Axiom* A.

A fundamental tool for the proof of the theorem will be a lemma (Lemma 3.1) essentially contained in the proof of a theorem of Jakobson [**2**, Theorem A].

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2. Preliminaries. Let $f \in N(S^1)$. A periodic point x of periodic n is called *hyperbolic, contracting* or *expanding* according as $|df^n(x)| \neq 1, < 1$ or > 1, respectively. Let $\Omega_c(f)$ denote the set of contracting periodic points of f. The set $\Omega(f) - \Omega_c(f)$ will be denoted by $\Omega_e(f)$.

Let $x \in \Omega_c(f)$. The *stable manifold* of x, $W^s(x)$, is defined by $W^s(x) = \{y\colon x \in \omega(y)\}$. Here $\omega(y)$ denotes the ω-limit set of the orbit $\{f^n(y)\}$. In general, $W^s(x)$ consists of countably many disjoint intervals. The interval containing x will be called the local *stable manifold* and is denoted by $W^s_l(x)$. The *stable manifold of f*,

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$\Delta(f)$, is defined by $\Delta(f) = \bigcup W^s(x)$, where the union is taken over all contracting periodic points x of f. We let $\Sigma(f) = S^1 - \Delta(f)$.

A point x is *eventually periodic* if some iterate of x is a periodic point of f. A point x is called *recurrent* if $x \in \omega(x)$.

3. Proof of the theorem. We start establishing some preliminary results.

The following lemma is a consequence of Theorem 3 and the Remark to Theorem 3 of [**2**].

LEMMA 3.1. *Let f be a C^2 nonsingular endomorphism of S^1 satisfying:*

(i) *All periodic points are hyperbolic;*

(ii) *f has a finite number (nonzero) of contracting periodic points.*

Then $\Sigma(f)$ is totally disconnected, and $f|_{\Sigma(f)}$ and $f|_{\Omega_e(f)}$ are topologically conjugate to subsemishifts of finite type.

Using this result it is possible to prove the following lemmas.

LEMMA 3.2. *Let $f \in N(S^1)$ be structurally stable of degree > 1, and let $\varepsilon > 0$. If $p \in \Omega_e(f)$ is a recurrent point of f, then there exists $q \in \mathrm{Per}(f)$ such that $|f^j(p) - f^j(q)| < \varepsilon$ for all $0 \leq j < m$, where m is the period of p.*

PROOF. We divide the proof of the lemma in two cases.

(1) $\Omega_c(f) \neq \emptyset$. By hypothesis f has all periodic points hyperbolic.

Choose a C^2 nonsingular endomorphism f_1 sufficiently close to f in the C^1 topology. By hypothesis f_1 and f are topologically conjugate. It follows from this that f_1 has all periodic points hyperbolic and $\Omega_c(f_1) \neq \emptyset$, because f has these properties and a conjugacy preserves hyperbolic periodic points and nonwandering points. By Szlenk's Lemma [**2**, Lemma 4], f_1 can be approximated in the C^1 topology by a C^2 nonsingular endomorphism f_2 satisfying the hypothesis of Lemma 3.1. Hence, $f_{2|\Omega_e(f_2)}$ is topologically conjugate to a subsemishift of finite type, and since f_2 and f are topologically conjugate, $f|_{\Omega_e(f)}$ is also topologically conjugate to a subsemishift of finite type. Therefore, there is a periodic point $q \in \Omega_e(f)$ such that $|f^j(p) - f^j(q)| < \varepsilon$ for all $0 \leq j \leq m$, where m is the period of q.

(2) $\Omega_c(f) = \emptyset$. By [**2**, Lemma 5], $\Omega(f) = S^1$. Hence by [**5**, Lemma 3.1], f is topologically conjugate to the expanding map $\delta_d \colon S^1 \to S^1$ defined by $\delta_d(z) = z^d$, where $d = \mathrm{degree} f$. Then the lemma follows from the fact that δ_d satisfies the property of the lemma and the uniform continuity of the conjugation.

LEMMA 3.3. *Let $f \in N(S^1)$ be structurally stable of degree > 1. If $p \in \Omega_e(f)$ is a recurrent point of f then there exists $n \in N$ such that $|df^n(p)| > 1$.*

PROOF. By the proof of Lemma 3.2, $\Omega_c(f)$ is finite.

Choose $0 < \varepsilon < 1$ and a compact neighborhood U_c of $\Omega_c(f)$ such that if g is ε close to f in the C^1 metric then g is topologically conjugate to f and $\Omega_c(g) \subset U_c$. By Lemma 3.2 and the fact that df is uniformly continuous, there exists a periodic point $q \in \Omega_e(f)$ such that $|df(f^l(p)) - df(f^l(q))| < \varepsilon/3$ for all $0 \leq l < n$, where n is the period of q.

Now, to prove the lemma we shall adapt techniques due to Franks [**1**] and Mañe [**3**]. Choose a number δ, $0 < \delta < \varepsilon/3$, such that if $I_l = \{x \in S^1 \mid |x - f^l(q)| < \delta\}$ then $I_l \cap U_c = \emptyset$ for all $0 \leq l < n$, and I_i and I_j are disjoint when $i \neq j$.

For every $l = 0, \ldots, n-1$, choose a C^∞ real valued function σ_l such that $0 \leq \sigma_l(x) \leq 1$, $\sigma_l(x) = 0$ if $x \in S^1 - I_l$, $\sigma_l(x) = 1$ if $|x - f^l(q)| \leq \delta/4$ and $|\sigma_l'(x)| < 2/\delta$ for all x. Let $g \in N(S^1)$ be defined by

$$g(x) = f(x) + \sum_{i=0}^{n-1} \sigma_i(x)\tau_i(x - f^i(q)),$$

where $\tau_i = df(f^i(p)) - df(f^i(q))$.

It is easy to see that $g(f^l(q)) = f(f^l(q))$ for all $0 \leq l \leq n$, and g is ε close to f in the C^1 metric, and therefore all periodic points of g are hyperbolic and $\Omega_c(g) \subset U_c$. These properties imply that $g^n(q) = q$ and q is expanding.

Moreover,

$$\begin{aligned} dg(f^l(q)) &= df(f^l(q)) + \sum_{i=0}^{n-1}[\sigma_i'(f^l(q))\tau_i(f^l(q) - f^i(q)) + \sigma_i(f^l(q))\tau_i] \\ &= df(f^l(q)) + \tau_l = df(f^l(p)). \end{aligned}$$

Hence,

$$\begin{aligned} dg^n(q) &= \prod_{i=0}^{n-1} dg(g^i(q)) = \prod_{i=0}^{n-1} dg(f^i(q)) \\ &= \prod_{i=0}^{n-1} df(f^i(p)) = df^n(p). \end{aligned}$$

Therefore, $|df^n(p)| > 1$ and the lemma is proved.

Now we shall prove the theorem by adapting a technique due to Mañe [**3**]. Let $f \in N(S^1)$ be structurally stable. If degree$f = 1$ the theorem follows from Peixoto's theorem [**6**]; so we assume degree$f > 1$. By [**5**, Corollary 2.4], $\Omega(f) = \overline{\mathrm{Per}(f)}$ and, by the proof of Lemma 3.2, $\Omega_c(f)$ is finite. Hence to show that f satisfies Axiom A, it is sufficient to show that $f|_{\Omega_e}(f)$ is expanding, i.e., there exist $k > 0$ and $\lambda > 1$ such that $|df^n(x)| \geq k\lambda^n$.

By compactness and f-invariance of $\Omega_e(f)$, this property is equivalent to showing that there exist $n \in \mathbf{N}$ and $c > 1$ such that $|df^n(x)| > c$ for all $x \in \Omega_e(f)$. By using the compactness of $\Omega_e(f)$ and the chain rule, it is easy to prove that this is equivalent to the fact that for every $x \in \Omega_e(f)$ there exists $n = n(x)$ such that $|df^n(x)| > 1$. Hence, everything is reduced to proving this condition.

By contradiction, suppose there exists $x \in \Omega_e$ such that $|df^n(x)| \leq 1$ for all $n \in N$. Then $f|_{\omega(x)}$ is not expanding because $\omega(x)$ is compact. Let $\mathcal{S}$ be the family of compact f-invariant subsets Σ of $\omega(x)$ such that $f|_\Sigma$ is not expanding. It is easy to see that if $\{\Sigma_\alpha \mid \alpha \in \mathcal{A}\} \subset \mathcal{S}$ satisfies $\Sigma'_\alpha \subset \Sigma''_\alpha$ or $\Sigma''_\alpha \subset \Sigma'_\alpha$ for all $\alpha', \alpha'' \in \mathcal{A}$ then $\bigcap_\alpha \Sigma_\alpha \in \mathcal{S}$. Hence by Zorn's lemma there exists $\Sigma \in \mathcal{S}$ such that $\Sigma' \in \mathcal{S}$ and $\Sigma' \subset \Sigma$ imply $\Sigma' = \Sigma$. Since $f|_\Sigma$ is not expanding there exists $y \in \Sigma$ such that $|df^n(y)| \leq 1$ for all $n \in N$. Then $\omega(y) \in \mathcal{S}$ which, together with the fact that $\omega(y) \subset \Sigma$, implies that $\omega(y) = \Sigma$. It follows that y is a recurrent point of f. But by Lemma 3.3, y is not a recurrent point. This contradiction proves the theorem.

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