NECESSARY CONDITIONS FOR STABILITY
OF NONSINGULAR ENDOMORPHISMS OF THE CIRCLE
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ABSTRACT. In this article we prove that Axiom A is a necessary condition for structural stability of $C^1$ nonsingular endomorphisms of the circle.

1. Introduction. We say that an endomorphism $f: S^1 \to S^1$ of the circle is nonsingular if $df$ is injective at each point of $S^1$. Let $N(S^1)$ be the space of $C^1$ nonsingular endomorphisms of $S^1$ endowed with the $C^1$ topology. $f$ in $N(S^1)$ is said to be structurally stable if it has a neighborhood $U$ such that any $g \in U$ is topologically conjugate to $f$; i.e., there exists a homeomorphism $h: S^1 \to S^1$ satisfying $hf = gh$. In [4] Z. Nitecki proves that Axiom A is a sufficient condition for a nonsingular endomorphism of $S^1$ to be structurally stable. Recall that $f \in N(S^1)$ satisfies Axiom A if

(a) $\text{Per}(f) = \Omega(f)$. Here $\text{Per}(f)$ denotes the set of all periodic points of $f$ and $\Omega(f)$ the set of nonwandering points of $f$, i.e., $x \in \Omega(f)$ if and only if for any neighborhood $U$ of $x$ there is an integer $n > 0$ such that $f^n(U) \cap U \neq \emptyset$.

(b) $\Omega(f)$ has a hyperbolic structure, i.e., $\Omega(f)$ decomposes into a disjoint union of two closed, invariant subsets $\Omega(f) = \Omega_c \cup \Omega_e$ such that there exist $k > 0$, $0 < \lambda < 1$, satisfying

$$|df^n(x)| \leq k\lambda^n \quad \text{for all } x \in \Omega_c,$$

and

$$|df^n(x)| \geq k\lambda^{-n} \quad \text{for all } x \in \Omega_e.$$

The purpose of this paper is to prove that the condition above is necessary for structural stability.

THEOREM. If $f \in N(S^1)$ is structurally stable then $f$ satisfies Axiom A.

A fundamental tool for the proof of the theorem will be a lemma (Lemma 3.1) essentially contained in the proof of a theorem of Jakobson [2, Theorem A].

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2. Preliminaries. Let $f \in N(S^1)$. A periodic point $x$ of periodic $n$ is called hyperbolic, contracting or expanding according as $|df^n(x)| \neq 1, < 1$ or $> 1$, respectively. Let $\Omega_c(f)$ denote the set of contracting periodic points of $f$. The set $\Omega_e(f) = \Omega(f) - \Omega_c(f)$ will be denoted by $\Omega_e(f)$.

Let $x \in \Omega_e(f)$. The stable manifold of $x$, $W^s(x)$, is defined by $W^s(x) = \{y : x \in \omega(y)\}$. Here $\omega(y)$ denotes the $\omega$-limit set of the orbit $\{f^n(y)\}$. In general, $W^s(x)$ consists of countably many disjoint intervals. The interval containing $x$ will be called the local stable manifold and is denoted by $W^s_f(x)$. The stable manifold of $f$,
\( \Delta(f) \), is defined by \( \Delta(f) = \bigcup W^s(x) \), where the union is taken over all contracting periodic points \( x \) of \( f \). We let \( \Sigma(f) = S^1 - \Delta(f) \).

A point \( x \) is eventually periodic if some iterate of \( x \) is a periodic point of \( f \). A point \( x \) is called recurrent if \( x \in \omega(x) \).

3. Proof of the theorem. We start establishing some preliminary results.

The following lemma is a consequence of Theorem 3 and the Remark to Theorem 3 of [2].

**Lemma 3.1.** Let \( f \) be a \( C^2 \) nonsingular endomorphism of \( S^1 \) satisfying:

(i) All periodic points are hyperbolic;
(ii) \( f \) has a finite number (nonzero) of contracting periodic points.

Then \( \Sigma(f) \) is totally disconnected, and \( f|_{\Sigma(f)} \) and \( f|_{\Omega_\epsilon(f)} \) are topologically conjugate to subsemishifts of finite type.

Using this result it is possible to prove the following lemmas.

**Lemma 3.2.** Let \( f \in N(S^1) \) be structurally stable of degree \( > 1 \), and let \( \epsilon > 0 \).
If \( p \in \Omega_\epsilon(f) \) is a recurrent point of \( f \), then there exists \( q \in \text{Per}(f) \) such that
\[ |f^j(p) - f^j(q)| < \epsilon \text{ for all } 0 \leq j < m, \]
where \( m \) is the period of \( p \).

**Proof.** We divide the proof of the lemma in two cases.

1. \( \Omega_\epsilon(f) \neq 0 \). By hypothesis \( f \) has all periodic points hyperbolic.
   Choose a \( C^2 \) nonsingular endomorphism \( f_1 \) sufficiently close to \( f \) in the \( C^1 \) topology. By hypothesis \( f_1 \) and \( f \) are topologically conjugate. It follows from this that \( f_1 \) has all periodic points hyperbolic and \( \Omega_\epsilon(f_1) \neq 0 \), because \( f \) has these properties and a conjugacy preserves hyperbolic periodic points and nonwandering points. By Szlenk’s Lemma [2, Lemma 4], \( f_1 \) can be approximated in the \( C^1 \) topology by a \( C^2 \) nonsingular endomorphism \( f_2 \) satisfying the hypothesis of Lemma 3.1. Hence, \( f_2|_{\Omega_\epsilon(f_2)} \) is topologically conjugate to a subsemishift of finite type, and since \( f_2 \) and \( f \) are topologically conjugate, \( f|_{\Omega_\epsilon(f)} \) is also topologically conjugate to a subsemishift of finite type. Therefore, there is a periodic point \( q \in \Omega_\epsilon(f) \) such that
\[ |f^j(p) - f^j(q)| < \epsilon \text{ for all } 0 \leq j < m, \]
where \( m \) is the period of \( q \).

2. \( \Omega_\epsilon(f) = 0 \). By [2, Lemma 5], \( \Omega(f) = S^1 \). Hence by [5, Lemma 3.1], \( f \) is topologically conjugate to the expanding map \( \delta_d: S^1 \to S^1 \) defined by \( \delta_d(z) = zd \), where \( d = \deg f \). Then the lemma follows from the fact that \( \delta_d \) satisfies the property of the lemma and the uniform continuity of the conjugation.

**Lemma 3.3.** Let \( f \in N(S^1) \) be structurally stable of degree \( > 1 \). If \( p \in \Omega_\epsilon(f) \) is a recurrent point of \( f \) then there exists \( n \in N \) such that \( |df^n(p)| > 1 \).

**Proof.** By the proof of Lemma 3.2, \( \Omega_\epsilon(f) \) is finite.
Choose 0 < \( \epsilon < 1 \) and a compact neighborhood \( U_\epsilon \) of \( \Omega_\epsilon(f) \) such that if \( g \) is \( \epsilon \) close to \( f \) in the \( C^1 \) metric then \( g \) is topologically conjugate to \( f \) and \( \Omega_\epsilon(g) \subset U_\epsilon \).
By Lemma 3.2 and the fact that \( df \) is uniformly continuous, there exists a periodic point \( q \in \Omega_\epsilon(f) \) such that
\[ |df^l(p) - df^l(q)| < \epsilon/3 \text{ for all } 0 \leq l < n, \]
where \( n \) is the period of \( q \).

Now, to prove the lemma we shall adapt techniques due to Franks [1] and Mañe [3]. Choose a number \( \delta, 0 < \delta < \epsilon/3 \), such that if \( I_i = \{ x \in S^1 | |x - f^i(q)| < \delta \} \) then \( I_i \cap U_\epsilon = \emptyset \) for all 0 < \( l < n \), and \( I_i \) and \( I_j \) are disjoint when \( i \neq j \).
For every \( l = 0, \ldots, n - 1 \), choose a \( C^\infty \) real valued function \( \sigma_l \) such that \( 0 \leq \sigma_l(x) \leq 1 \), \( \sigma_l(x) = 0 \) if \( x \in S^1 - I_l \), \( \sigma_l(x) = 1 \) if \( |x - f^l(q)| \leq \delta/4 \) and \( |\sigma'_l(x)| < 2/\delta \) for all \( x \). Let \( g \in N(S^1) \) be defined by

\[
g(x) = f(x) + \sum_{i=0}^{n-1} \sigma_i(x)\tau_i(x - f^i(q)),
\]

where \( \tau_i = df(f^i(p)) - df(f^i(q)) \).

It is easy to see that \( g(f^l(q)) = f(f^l(q)) \) for all \( 0 \leq l \leq n \), and \( g \) is \( \epsilon \) close to \( f \) in the \( C^1 \) metric, and therefore all periodic points of \( g \) are hyperbolic and \( \Omega_c(g) \subseteq U_c \).

These properties imply that \( g^n(q) = q \) and \( q \) is expanding.

Moreover,

\[
dg(f^l(q)) = df(f^l(q)) + \sum_{i=0}^{n-1} [\sigma'_i(f^l(q))\tau_i(f^l(q) - f^i(q)) + \sigma_i(f^l(q))\tau_i]
\]

\[
= df(f^l(q)) + \tau_l = df(f^l(p)).
\]

Hence,

\[
dg^n(q) = \prod_{i=0}^{n-1} dg(f^i(q)) = \prod_{i=0}^{n-1} df(f^i(q))
\]

\[
= \prod_{i=0}^{n-1} df(f^i(p)) = df^n(p).
\]

Therefore, \( |df^n(p)| > 1 \) and the lemma is proved.

Now we shall prove the theorem by adapting a technique due to Mañé [3]. Let \( f \in N(S^1) \) be structurally stable. If degree \( f = 1 \) the theorem follows from Peixoto's theorem [6]; so we assume degree \( f > 1 \). By [5, Corollary 2.4], \( \Omega(f) = \text{Per}(f) \) and, by the proof of Lemma 3.2, \( \Omega_c(f) \) is finite. Hence to show that \( f \) satisfies Axiom A, it is sufficient to show that \( f|_{\Omega_c(f)} \) is expanding, i.e., there exist \( k > 0 \) and \( \lambda > 1 \) such that \( |df^n(x)| > k\lambda^n \).

By compactness and \( f \)-invariance of \( \Omega_c(f) \), this property is equivalent to showing that there exist \( n \in \mathbb{N} \) and \( c > 1 \) such that \( |df^n(x)| > c \) for all \( x \in \Omega_c(f) \). By using the compactness of \( \Omega_c(f) \) and the chain rule, it is easy to prove that this is equivalent to the fact that for every \( x \in \Omega_c(f) \) there exists \( n = n(x) \) such that \( |df^n(x)| > 1 \). Hence, everything is reduced to proving this condition.

By contradiction, suppose there exists \( x \in \Omega_c \) such that \( |df^n(x)| \leq 1 \) for all \( n \in \mathbb{N} \). Then \( f|_{\omega(x)} \) is not expanding because \( \omega(x) \) is compact. Let \( S \) be the family of compact \( f \)-invariant subsets \( \Sigma \) of \( \omega(x) \) such that \( f|_{\Sigma} \) is not expanding. It is easy to see that if \( \{ \Sigma_\alpha \mid \alpha \in \mathcal{A} \} \subseteq S \) satisfies \( \Sigma'_\alpha \subseteq \Sigma''_\alpha \) or \( \Sigma''_\alpha \subseteq \Sigma'_\alpha \) for all \( \alpha', \alpha'' \in \mathcal{A} \) then \( \bigcap \Sigma_\alpha \subseteq S \). Hence by Zorn's lemma there exists \( \Sigma \in S \) such that \( \Sigma' \subseteq S \) and \( \Sigma' \subseteq \Sigma \) imply \( \Sigma' = \Sigma \). Since \( f|_{\Sigma} \) is not expanding there exists \( y \in \Sigma \) such that \( |df^n(y)| \leq 1 \) for all \( n \in \mathbb{N} \). Then \( \omega(y) \in S \) which, together with the fact that \( \omega(y) \subseteq \Sigma \), implies that \( \omega(y) = \Sigma \). It follows that \( y \) is a recurrent point of \( f \). But by Lemma 3.3, \( y \) is not a recurrent point. This contradiction proves the theorem.
REFERENCES


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