A CHAOTIC FUNCTION POSSESSING A SCRAMBLED SET WITH POSITIVE LEBESGUE MEASURE

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ABSTRACT. A continuous function, chaotic in the sense of Li and Yorke, is constructed which possesses a scrambled set of positive Lebesgue measure.

Introduction. A continuous function \( f: I \to I \), where \( I \) is a real finite interval, is called chaotic (in the sense of Li and Yorke [1]) provided there exists an uncountable set \( S \subset I \) such that for any \( x, y \in S \), \( x \neq y \), and \( p \) any periodic point of \( f \):

\[
\begin{align*}
(1) & \quad \limsup_{n \to \infty} |f^n(x) - f^n(y)| > 0, \\
(2) & \quad \liminf_{n \to \infty} |f^n(x) - f^n(y)| = 0, \\
(3) & \quad \limsup_{n \to \infty} |f^n(x) - f^n(p)| > 0,
\end{align*}
\]

where \( f^n \) is the \( n \)th iterate of \( f \). We call any such set \( S \) a scrambled set of \( f \). We call a scrambled set \( E \) extremally scrambled, iff for any \( x, y \in E \), \( x \neq y \), and \( p \) any periodic point of \( f \):

\[
\begin{align*}
(4) & \quad \limsup_{n \to \infty} |f^n(x) - f^n(y)| = \text{diam}(I), \\
(5) & \quad \liminf_{n \to \infty} |f^n(x) - f^n(y)| = 0, \\
(6) & \quad \limsup_{n \to \infty} |f^n(x) - f^n(p)| \geq \text{diam}(I)/2.
\end{align*}
\]

J. Smital has shown that \( f: [0, 1] \to [0, 1] \) where

\[
f(x) = \begin{cases} 
2x, & 0 \leq x \leq 1/2, \\
2 - 2x, & 1/2 < x \leq 1,
\end{cases}
\]

has an extremally scrambled set of Lebesgue outer measure 1 [2]. This set is not Lebesgue measurable, and Smital's construction of it makes use of the continuum hypothesis.

In the literature there is no known chaotic function possessing a scrambled set of positive Lebesgue measure. Here, without using the continuum hypothesis, we construct a continuous chaotic function \( f: [0, 1] \to [0, 1] \), along with an extremally scrambled set \( E \) of Lebesgue measure \( 1/8 \).
First we consider the class $\mathcal{G}$ of all continuous functions $g: [0,1] \to [0,1]$ such that $g(x) = 3x$ for $x \in [0, \frac{1}{3}]$ and $g(x) = 3x - 2$ for $x \in [\frac{2}{3}, 1]$, and find a set $K \subset (0, \frac{1}{3})$ which is an extremally scrambled set of every $g \in \mathcal{G}$. Any such $g$ is clearly chaotic. Then we construct a fat Cantor set $E \subset [\frac{3}{8}, \frac{5}{8}]$ with Lebesgue measure $\frac{1}{8}$. Finally we choose a particular $f \in \mathcal{G}$ which, when restricted to $E$, is a monomorphism (actually a homeomorphism) from $E$ to $K$. Since $K$ is an extremally scrambled set of $f$, $E$ is one also.

**Preliminaries.** Let $N$ denote the natural numbers. Let $\Omega = \{0,1\}^N$ be the space of all one-sided sequences of two symbols, along with the dictionary ordering relation $<$, and the topology of coordinatewise convergence. The shift $\sigma$ on $\Omega$ is defined by $(\sigma \omega)_k = \omega_{k+1}$, $k \in N$.

Let the map $\varphi: \Omega \to [0,1]$ be defined by
$$\varphi(\omega) = 2 \sum_{k=1}^{\infty} \omega_k 3^{-k}.$$ 
Then the image under $\varphi$ of $\Omega$ is the usual “middle thirds” Cantor set $C$, and $\varphi$ is an order preserving homeomorphism from $\Omega$ to $C$. Also, for any $g \in \mathcal{G}$, we have
$$g(\varphi(\omega)) = 3\varphi(\omega) - 2\omega_1 = 2 \sum_{k=1}^{\infty} \omega_{k+1} 3^{-k} = \varphi(\sigma \omega).$$
And, for $n \in N$, $g^n(\varphi(\omega)) = \varphi(\sigma^n \omega)$.

The following simple lemma will prove useful.

**Lemma.** (i) If $g \in \mathcal{G}$, $\alpha, \beta \in \Omega$, $\alpha_{n+j} = \beta_{n+j}$ for $j = 1, \ldots, k$, then
$$|g^n(\varphi(\alpha)) - g^n(\varphi(\beta))| \leq 3^{-k}.$$
(ii) If $g \in \mathcal{G}$, $\alpha \in \Omega$, $\alpha_{n+j} = 0$ ($\alpha_{n+j} = 1$), for $j = 1, \ldots, k$, then
$$g^n(\varphi(\alpha)) \leq 3^{-k}(g^n(\varphi(\alpha)) \geq 1 - 3^{-k}).$$

**Proof.** The lemma clearly follows from the fact that
$$g^n(\varphi(\omega)) = \varphi(\sigma^n \omega) = 2 \sum_{j=1}^{\infty} \omega_{n+j} 3^{-j}.$$ 

I. We construct a set $K \subset C$, which is an extremally scrambled set of every $g \in \mathcal{G}$. The method used here is similar to that used by M. Osikawa and Y. Oono to construct a scrambled set [3].

Let $r: N \to N$ be defined by
$$r(k) = \inf \left\{ l \in N \mid k \leq \sum_{j=1}^{l} j^2 + 2j \right\}$$
and let $s: N \to N$ be defined by
$$s(k) = \sup \left\{ l \in N \cup \{0\} \mid l + 1 \leq \left( k - \sum_{j=1}^{r(k)-1} j^2 + 2j \right)/r(k) \right\}$$
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where $\sum_{j=1}^{0} j^2 + 2j$ is defined to be 0. Then define $Z: \Omega \to \Omega$ by

$$(Z(\omega))_k = \begin{cases} 
0 & \text{if } s(k) = 0, \\
1 & \text{if } s(k) = 1, \\
\omega_{s(k)-1} & \text{if } s(k) \geq 2,
\end{cases}$$

i.e., $Z(\omega) = 01\omega_10011\omega_1\omega_1\omega_2\omega_2\cdots$. Let $K$ be the set $\varphi(Z(\Omega))$.

The map $Z: \Omega \to Z(\Omega)$ is an order preserving homeomorphism, and since $\varphi: \Omega \to C$ is also, we see that $K$ is homeomorphic to $C$. This fact will be important in §III, where we construct a homeomorphism from the fat Cantor set $E$ of §II to $K$.

PROPOSITION. $K$ is an extremally scrambled set of every $g \in \mathcal{G}$.

PROOF. Let $x, y \in K$, $x < y$, and $p$ be any periodic point of $g$ with period $q$. It suffices to show that equations (4)-(6) hold with $g$ in place of $f$.

Let $\alpha = \varphi^{-1}(x)$ and $\beta = \varphi^{-1}(y)$. Then by the construction of $K$ we have, for any $k \in \mathbb{N}$, infinitely many $n \in \mathbb{N}$ such that $\alpha_{n+j} = \beta_{n+j} = 0$ for $j = 1, \ldots, k$. Thus, by (i) of the Lemma, (5) is satisfied. Since $x < y$, we also have infinitely many $n \in \mathbb{N}$ such that $\alpha_{n+j} = 0$ and $\beta_{n+j} = 1$, for $j = 1, \ldots, k$. Thus, by (ii) of the Lemma, (4) is satisfied. We also have infinitely many $n \in \mathbb{N}$ such that $\alpha_{n+j} = 0$ ($\alpha_{n+j} = 1$) for $j = 1, \ldots, k + q$ so that

$$\limsup_{n \to \infty} |g^n(x) - g^n(p)| \geq \max\{g^n(p) \mid n \in \mathbb{N}\}$$

and

$$\limsup_{n \to \infty} |g^n(x) - g^n(p)| \geq \max\{1 - g^n(p) \mid n \in \mathbb{N}\}$$

holds. Therefore,

$$\limsup_{n \to \infty} |g^n(x) - g^n(p)| \geq 1/2$$

holds.

II. Here we constructed a fat Cantor set $E \subset [\frac{3}{8}, \frac{5}{8}]$ so that $m(E) = \frac{1}{8}$ [4, 5]. The set $E$ will be the countable intersection of a nested sequence of compact sets $\{E_n\}$ where each $E_n$ is the union of $2^n$ disjoint closed intervals, exactly two of these intervals being contained in each one of the $2^{n-1}$ disjoint closed intervals of $E_{n-1}$. Also, the diameters of the intervals comprising $E_n$ go uniformly to 0 as $n$ goes to $\infty$.

Let $\Gamma_n = \{0, 1\}^n$ denote the set of all two symbols of length $n$, and $\gamma_k$ is the $k$th coordinate of $\gamma \in \Gamma_n$. There are $2^n$ elements of $\Gamma_n$.

The sine of $\pi/6$ can be written as an infinite product [6] as follows,

$$\sin \frac{\pi}{6} = \frac{\pi}{6} \prod_{l=1}^{\infty} \left(1 - \frac{1}{36l^2}\right) = \frac{1}{2}.$$ 

Let $E_0 = [\frac{3}{8}, \frac{5}{8}]$ and let $E_n$ be the union of the $2^n$ disjoint closed intervals $I(\gamma)$, where $\gamma \in \Gamma_n$. The right-hand endpoint of $I(\gamma)$ is

$$b(\gamma) = \frac{5}{8} - \frac{1}{4} \sum_{k=1}^{n} \gamma_k 2^{-k} \frac{\pi}{6} \prod_{l=1}^{6} \left(1 - \frac{1}{36l^2}\right).$$
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and the left-hand endpoint of \( I(\gamma) \) is

\[
a(\gamma) = b(\gamma) - \frac{1}{4} 2^{-n} \pi \frac{\pi}{6} \prod_{l=1}^{n+1} \left( 1 - \frac{1}{36l^2} \right).
\]

It follows that if \( n \geq 2, \gamma \in \Gamma_n, \lambda \in \Gamma_{n-1} \), then either \( I(\gamma) \) and \( I(\lambda) \) are disjoint, or \( I(\gamma) \) is contained in \( I(\lambda) \). The latter occurs iff \( \gamma_k = \lambda_k \) for \( k = 1, \ldots, n-1 \). The Lebesgue measure of \( E_n \) is

\[
\frac{1}{4} \pi \frac{\pi}{6} \prod_{l=1}^{n+1} \left( 1 - \frac{1}{36l^2} \right)
\]

and, since \( E_n \cap E = \bigcap_{n} E_n \), we have

\[
m(E) = \lim_{n \to \infty} E_n = \frac{1}{8}.
\]

A point is in \( E \) iff it is a limit point of the right-hand endpoints of the intervals comprising the sets \( E_n \). So, the map \( \psi: \Omega \to E \) defined by

\[
\psi(\omega) = \frac{5}{8} - \frac{1}{4} \sum_{k=1}^{\infty} \omega_k 2^{-k} \pi \frac{\pi}{6} \prod_{l=1}^{k} \left( 1 - \frac{1}{36l^2} \right)
\]

is an order reversing homeomorphism.

III. We now choose a particular \( f \in \mathcal{G} \) so that \( f \) restricted to \( E \) is a homeomorphism (monomorphism is all that is necessary) from \( E \) to \( K \). Then for any \( x, y \in E, x \neq y, p \) any periodic point of \( f \), we have \( f(x), f(y) \in K, f(x) \neq f(y), f(p) \) a periodic point of \( f \). Thus \( E \) will be an extremely scrambled set of \( f \).

Let \( f(t) = \varphi(Z(\psi^{-1}(t))) \) for \( t \in E \). Since \( f \in \mathcal{G} \), we have \( f(t) = 3t \) for \( t \in [0, \frac{1}{3}] \) and \( f(t) = 3t - 2 \) for \( t \in \left[ \frac{2}{3}, 1 \right] \). We now define \( f(t) \) for \( t \in E \cap \left( \frac{1}{3}, \frac{2}{3} \right) \) to be a linear interpolation of the values of \( f \) on the nearest points of \( E \cup \left\{ \frac{1}{3} \right\} \cup \left\{ \frac{2}{3} \right\} \) to \( t \). More precisely, for \( t \in E \cap \left( \frac{1}{3}, \frac{2}{3} \right) \), we define

\[
f(t) = f(t_i) + \frac{(t - t_i)}{(t_r - t_i)} (f(t_r) - f(t_i))
\]

where \( t_i = \sup\{t' \in E \mid t' < t\} \cup \left\{ \frac{1}{3} \right\} \) and \( t_r = \inf\{t' \in E \mid t' > t\} \cup \left\{ \frac{2}{3} \right\} \).

**Proposition.** The function \( f \) is continuous and \( K \) is the homeomorphic image under \( f \) of \( E \).

**Proof.** It suffices to show that \( f \) is continuous, strictly decreasing on \( [\frac{1}{3}, \frac{2}{3}] \), and \( f(E) = K \).

The image under \( f \) of \( E \) is the image under \( \varphi \circ Z \) of \( \psi^{-1}(E) \), but \( \psi^{-1}(E) = \Omega \), so

\[
f(E) = \varphi(Z(\psi^{-1}(E))) = \varphi(Z(\Omega)) = K.
\]

The function \( f \) is continuous and strictly decreasing on \( E \cup \left\{ \frac{1}{3} \right\} \cup \left\{ \frac{2}{3} \right\} \), since \( f \left( \frac{1}{3} \right) = 1, f \left( \frac{2}{3} \right) = 0, f(E) = K \subset (0, \frac{1}{3}) \), and also \( \psi^{-1} \) is continuous and order reversing and both \( \varphi \) and \( Z \) are continuous and order preserving. On the rest of \( [\frac{1}{3}, \frac{2}{3}] \), \( f \) is just contained by linear interpolation, so \( f \) is continuous and strictly decreasing on \( [\frac{1}{3}, \frac{2}{3}] \).

**Conclusion.** The \( f \) constructed in the third part is a chaotic function possessing an extremely scrambled set \( E \) of Lebesgue measure \( \frac{1}{8} \).

**Remark.** It is easy to find a chaotic function \( h: [0,1]^r \to [0,1]^r \) which has a scrambled set of positive \( r \) dimensional Lebesgue measure \( m_r \). Let \( h \) be defined by

\[
h(x_1, \ldots, x_r) = (f(x_1), \ldots, f(x_r)).
\]
Then $E^r$ is a scrambled set of $h$ and $m_r(E^r) = (\frac{1}{3})^r$. In fact, for any $x, y \in E^r, x \neq y$, $p$ any periodic point of $h$, we have

$$\limsup_{n \to \infty} |h^n(x) - h^n(y)| \geq 1,$$

$$\liminf_{n \to \infty} |h^n(x) - h^n(y)| = 0,$$

$$\limsup_{n \to \infty} |h^n(x) - h^n(p)| \geq \sqrt{r}/2.$$ 

References


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