ABSTRACT. It is shown that every \( \sigma \)-additive \( \sigma \)-finite invariant measure on an abelian group has a proper \( \sigma \)-additive invariant extension.

We consider \( \sigma \)-finite countably additive measures which vanish on points and are nonidentically zero. Throughout this paper the word "measure" will mean a measure enjoying all the above properties. A measure \( m \) defined on a \( \sigma \)-algebra \( S \) of subsets of \( X \) is called invariant with respect to a group \( G \) of bijections of \( X \) if for any \( T \in G \) and \( A \in S \) the image \( T^*(A) \) is an element of \( S \) and \( m(T^*(A)) = m(A) \). A measure \( m \) defined on a \( \sigma \)-algebra of subsets of a group \( G \) is called invariant if it is invariant with respect to left translations.

Sierpiński (quoted in Szpilrajn [7]) asked whether there exists in Euclidean \( n \)-dimensional space \( E^n \) a maximal extension of the Lebesgue measure invariant with respect to the group of isometries of \( E^n \). Hulanicki [2] proved that if \( |X| \) is less than the first real-valued measurable cardinal, \( |G| \leq |X| \), and \( m \) is a measure on \( X \) invariant with respect to \( G \) and vanishing on sets of cardinality \( < |X| \), then there exists a proper extension of \( m \) invariant with respect to \( G \). Thus he solved Sierpiński's problem under additional set theoretic assumptions. Harazišvili [1] gave a negative answer to this question for \( n = 1 \) without any extra hypotheses. He also proved that there is no maximal measure invariant with respect to translations on any Euclidean space. In other words the group of translations of \( E^n \) does not carry maximal invariant measures. Our theorem is a generalisation of the above result.

THEOREM. Every invariant measure on an abelian group \((G, +)\) has a proper invariant extension.

PROOF. We start with the following lemma, essentially due to Szpilrajn [7]. The easy proof is left to the reader.

LEMMA. Let \( m \) be an invariant measure on \( G \). If there exists a set \( E \subset G \) such that:
1. \( E \) is not a set of \( m \) measure zero.
2. For every sequence \( \{g_n: n \in \omega\} \) of elements of \( G \), there exists a sequence \( \{h_\alpha: \alpha < \omega_1\} \) of elements of \( G \) such that for distinct \( \alpha, \beta \),
\[
m\left( h_\alpha + \bigcup_{n \in \omega} (g_n + E) \right) \cap \left( h_\beta + \bigcup_{n \in \omega} (g_n + E) \right) = \emptyset,
\]
then the measure \( m \) has a proper invariant extension.
Hence it suffices to show a set $E$ with the above properties. Without loss of
generality we assume that $m$ is a complete measure (i.e. subsets of measure zero
sets are measurable).

**Case 1.** Additive groups of linear spaces over a countable field (cf. Harazišvili [1]
and Pelc [4]). Let $V$ be a linear space over a countable field $K$ and $m$ any measure
on $V$ invariant with respect to addition. Fix a linear basis $\mathcal{B} = \{V_\alpha: \alpha < \kappa\}$ of $V$
over $K$ and let $V_n$ denote the set of those elements of $V$ which have $n$ summands
in the basis $\mathcal{B}$ representation.

Hence $V = \bigcup_{n \in \omega} V_n$ and there exists the least number $n_0$ for which $V_{n_0}$ does
not have measure 0. We claim that $V_{n_0}$ also satisfies condition 2 of the lemma.

Let $\{g_n: n \in \omega\}$ be a countable sequence of elements of $V$ and

$$D = \bigcup_{n \in \omega} (g_n + V_{n_0}).$$

As $\{h_\alpha: \alpha < \omega_1\}$ for the lemma take any subset of $\mathcal{B}$ of cardinality $\omega_1$ whose
elements do not appear in the $\mathcal{B}$-representation of $g_i - g_j$ where $(i, j) \in \omega \times \omega$.
Then $w = h_\alpha + g_i + w_1 = h_\beta + g_j + w_2$, where $w_1$ and $w_2$ are in $V_{n_0}$, if and only if
$g_i - g_j = h_\beta - h_\alpha + w_2 - w_1$. Since $h_\beta$ and $h_\alpha$ are not used in the $\mathcal{B}$-representation
of $g_i - g_j$ and they are distinct, then either $w_1 = kh_\alpha + w'$ or $w_2 = kh_\alpha + w'$ for
some $k \in K$ and $w'$ in $V_{n_0-1}$. Hence $w = k'h_\alpha + g_i + w'$ or $w = h_\beta + g_j + k'h_\alpha + w'$
for some $k' \in K$ and $w' \in V_{n_0-1}$ so that for $\alpha \neq \beta$ the set $(h_\alpha + D) \cap (h_\beta + D)$ is
a subset of a countable union of translations of $V_{n_0-1}$. Therefore $(h_\alpha + D) \cap (h_\beta + D)$
has $m$ measure zero. Hence the set $V_{n_0}$ satisfies the conditions of the lemma.

**Case 2.** Torsion-free abelian groups. Let $G$ be a torsion-free abelian group.
There exists a homomorphic embedding of $G$ into the additive group of a linear
space $V$ over the field $Q$ of rationals such that a certain basis $\mathcal{B} = \{v_\alpha: \alpha < \kappa\}$ of $V$
consists of elements of $G$. Let $m$ be any invariant measure on $G$.

For any finite sequence $s = (q_1, \ldots, q_n)$ of nonzero rationals let $V_s$ be the set of
elements of $V$ of the form $q_1v_{\alpha_1} + \cdots + q_nv_{\alpha_n}$ where $\alpha_1 > \cdots > \alpha_n$ and $v_{\alpha_i} \in \mathcal{B}$.
Let $s_0 = (r_1, \ldots, r_n)$ be a sequence for which the set $E = G \cap V_{s_0}$ is not a set of
$m$ measure 0. In order to check that $E$ also satisfies condition 2 of the lemma,
let $\{g_n: n \in \omega\}$ be any sequence of elements in $G$. Take any uncountable set of
elements $w_\alpha$ of $\mathcal{B}$ which do not appear in the $\mathcal{B}$-representation of any element $g_n$.
Let $k$ be a natural number different from all $r_i$, $r_i - r_j$ ($i, j \leq n$) and $h_\alpha = kw_\alpha$ for
$\alpha < \omega_1$. We claim that

$$\left[ h_\alpha + \bigcup_{n \in \omega} (g_n + E) \right] \cap \left[ h_\beta + \bigcup_{n \in \omega} (g_n + E) \right] = \emptyset.$$

Indeed, suppose $x$ is an element of the set on the left side. Then

$$x = kw_\alpha + g_n + r_1v_{\alpha_1} + \cdots + r_nv_{\alpha_n} = kw_\beta + g_m + r_1v_{\beta_1} + \cdots + r_nv_{\beta_n}.$$

Since $\alpha \neq \beta$ and $w_\alpha$, $w_\beta$ do not appear in the representation of $g_n$, $g_m$, we get that
either $k = r_i$ or $k + r_i = r_j$ for some $i, j \leq n$, contradiction.

**Case 3.** Arbitrary groups. Let $G$ be an arbitrary abelian group and $m$ an
invariant measure on $G$. By $H$ denote the torsion subgroup of $G$. If $m(H) = 0$ we
define a measure $m_1$ on $G/H$, putting $m_1(\{a + H: a \in A\}) = m(\bigcup_{a \in A}(a + H))$.
for $A \subset G$ such that $\bigcup_{a \in A} (a + H)$ is $m$-measurable. The measure $m_1$ is clearly invariant (and vanishes on points since $m(H) = 0$). The group $G/H$ is torsion-free and, hence, by Case 2 there exists a set $E_1 \subset G/H$ satisfying both conditions from the lemma for $G/H$ and $m_1$. It is not hard to see that the set $E = \bigcup E_1$ satisfies the conditions from the lemma for $G$ and $m$.

If $H$ is not a set of $m$ measure 0 then let $H_n$ (for $n \geq 1$) denote the subgroup of $H$ consisting of those elements whose orders divide $n$. Clearly $H = \bigcup_{n \geq 1} H_n$ and let $n_0$ be the least natural number for which $H_{n_0}$ is not a set of $m$ measure 0. We will prove the existence of a subset of $H_{n_0}$ satisfying the conditions of our lemma by induction on the number of prime divisors of $n_0$ (counting multiple divisors many times). If $k = 1$ then $n_0$ is prime and $H_{n_0}$ is the additive group of a linear space over the field $F_{n_0}$. Next we proceed as in Case 1 and show that the set constructed there is as required (for $G$ and $m$).

Suppose that for $n_0$ having $k$ prime divisors there exists a set $E \subset H_{n_0}$ satisfying the lemma. Now let $n_0 = p_1 \cdots p_{k+1}$ ($p_i$-primes, $k \geq 1$) and let $H'$ be the subgroup of $H_{n_0}$ consisting of elements of order $p_1$. Since $m(H') = 0$, we can define an invariant measure $m'$ on $G/H'$ just as before. $H_{n_0}/H'$ is a subgroup of $G/H'$ all of whose elements have orders dividing the number $p_2 \cdots p_{k+1}$. By definition $H_{n_0}/H'$ is not a set of $m'$ measure 0. Hence by the inductive hypothesis there exists a set $E' \subset H_{n_0}/H'$ which satisfies the conditions of the lemma for the group $G/H'$ and measure $m'$. It is easy to see that set $E = \bigcup E'$ is now good for $G$ and $m$, which finishes the proof in the general case.

REFERENCES

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