ON A PROBLEM OF HELLERSTEIN, SHEN AND WILLIAMSON

A. HINKKANEN AND J. ROSSI

Abstract. Suppose that \( f \) is a nonentire transcendental meromorphic function, real on the real axis, such that \( f \) and \( f' \) have only real zeros and poles, and \( f' \) omits a nonzero value. Confirming a conjecture of Hellerstein, Shen and Williamson, it is shown that then \( f \) is essentially \( f(z) = \tan z - Bz - C \) for suitable values of \( B \) and \( C \).

The purpose of this note is to prove the following theorem.

Theorem 1. Suppose that \( f \) is a nonentire, transcendental meromorphic function, real on the real axis with only real poles, and that the zeros of \( f \) and \( f' \) are real. If \( f' \) omits a nonzero value, then the omitted value is real and

\[
(1) \quad f(z) = A \left[ \tan(az + b) - B(az + b) - C \right],
\]

where \( A, B, C, a, b \) are real, \( A \neq 0 \neq a, B \geq 1, \) and \( C \) has the following property. Let \( n \) be an integer such that

\[
-\frac{1}{2}B\pi < C_0 = C + nB\pi \leq \frac{1}{2}B\pi.
\]

Then with \( \beta = (\sqrt{B - 1} + |C_0|)/B, \) we have \( \beta < \frac{1}{2}\pi \) and \( \tan^2 \beta \leq B - 1. \) Further, the zeros of \( f'' \) are real.

Theorem 1 is connected to the problem of determining all meromorphic functions \( f \) with only real poles such that \( f, f' \) and \( f'' \) have only real zeros. By a result of Hellerstein, Shen and Williamson [4, Theorems 1 and 2], the only remaining open case is when \( f \) is real (i.e. \( f(z) \) is real or \( \infty \) if \( z \) is real) and not entire. For such transcendental functions \( f \) the only known examples are given by (1) and

\[
(2) \quad f(z) = A \tan(az + b) + B,
\]

where \( A, b, a, b \) are real and \( A \neq 0 \neq a. \) For \( f \) given by (1) (by (2)), \( f' \) omits a nonzero value (omits zero). Hellerstein, Shen and Williamson announced [5] that if \( f \) is a real, transcendental, nonentire solution to the problem and \( f' \) omits zero, then \( f \) is given by (2). Theorem 1 confirms their conjecture that if the value omitted by \( f' \) is nonzero instead, then \( f \) is given by (1). We note that they only had (1) with \( B = 1, C = 0. \) Here we do not need the assumption that the zeros of \( f'' \) are real.

Received by the editors July 20, 1983.

1980 Mathematics Subject Classification. Primary 30D30; Secondary 30D35.

1Research supported by the Osk. Huttunen Foundation, Helsinki.

2Research supported by a NATO Postdoctoral Fellowship.
To prove Theorem 1, we first note that if $f'$ omits $a, a \neq 0$, then the equations $f = 0, f = \infty$ and $f' = a$ have only real solutions. Hence by a theorem of Edrei [2] the order of $f$ does not exceed one. We can write
\begin{equation}
(3) \quad f'(z) = \alpha + 1/g(z),
\end{equation}
where $g$ is a transcendental entire function of order at most one with only real zeros and with at least one zero. Each zero of $g$ has order two at least.

We show that $\alpha$ is real. If not, then $f' = h/k$, where $h$ and $k$ are real entire functions of order one at most, with no common zeros and only real zeros. By (3), $h - ak = k/g$ is entire, not real, nonvanishing and of order one at most. Thus $(h - ak)(z) = Ae^{Bz}$ and $(h' - ak')(z) = ABe^{Bz}$, $A \neq 0$, where not both $A$ and $B$ are real. Since $g$ has only multiple zeros, so does $k$, and so we can find a real $X$ such that $k(X) = k'(X) = 0$. Since $h$ is real, so are $Ae^{BX}$ and $ABe^{BX}$. This means both $B$ and $A$ are real, which is a contradiction. And so $\alpha$ and hence $g$ are real.

Next we show that all the zeros of $g$ have order two. If not, let $X_0$ be a (real) zero of $g$ of order $k \geq 3$. Since $f'$ has only real zeros and its order does not exceed one, the function
\begin{equation}
(4) \quad g(z) + a^{-1} = a^{-1} + A_k(z - X_0)^k + \cdots
\end{equation}
has only real zeros and thus belongs to the Laguerre-Polya class $U_0$ (cf. [3 pp. 227, 228]). By a result of Polya and Schur [6, pp. 104, 121], such a function cannot have two or more successive vanishing coefficients between two nonvanishing ones in its Taylor series about a real point. This contradicts (4), so that all zeros of $g$ have order two. Hence all poles of $f$ are simple, and $g = h^2$ for some real entire function $h$ of order one at most, belonging to the Laguerre-Polya class.

The residue of $f'$, i.e. the residue of $1/h^2$, must vanish at every zero $X_0$ of $h$. This implies that $h''(X_0) = 0$. Hence $h''/h = k$, say, is entire, since the zeros of $h$ are simple. The lemma of the logarithmic derivative and the fact that $h$ has finite order imply that $k$ is a polynomial. Since the order of $h$ is at most one, well-known results [7, pp. 68–70] (also cf. [1, Theorem 1]) imply that $k$ is a constant. Since $h$ is transcendental, real and in $U_0$ with at least one zero, it follows that $h(z) = A_1 \cos(A_2 z + A_3)$ for some real $A_1, A_2, A_3$ with $A_1 \neq 0 \neq A_2$.

Now integration of (3) gives
\begin{equation}
f(z) = A[\tan(az + b) - B(az + b) - C]
\end{equation}
for real $A, B, C, a, b$ with $A \neq 0 \neq a, B \neq 0$. The zeros of $f''$ are then real, whereas $f^{(3)}$ has nonreal zeros. The zeros of $f'$ are real if and only if $B \geq 1$, and $f'$ omits $\alpha = -aAB$. (To see this note that $\tan z$ is real if and only if $z$ is real, and $\tan^2 z \neq -1$ for all $z$.)

To complete the proof of Theorem 1, it remains to show that the zeros of
\begin{equation}
f(z) = \tan(z - n\pi) - B(z - n\pi) - C_0,
\end{equation}
are real if and only if $C$ is admissible, i.e. has the property mentioned in Theorem 1. Since
we may assume that \( C = C_0 \). Then since \(-B\pi/2 < C \leq B\pi/2\) the reader may verify that \( f \) has

(i) exactly one zero in \( I_n, n = \pm 1, \pm 2, \ldots \),

(ii) exactly three (one) zeros in \( I_0 \),

if \( C \) is admissible (not admissible), where

\[
I_n = \left[ (2n - 1)\frac{\pi}{2}, (2n + 1)\frac{\pi}{2} \right].
\]

Now

\[
f(z) = \frac{-G'(z)}{\cos z} \exp \left[ -\frac{Bz^2}{2} - Cz \right],
\]

where

\[
G(z) = \cos z \exp(Bz^2/2 + Cz).
\]

By Rolle's theorem, \( G' \) has at least one zero in each of the \( I_n \). But since \( B > 0 \),
\( G \in U_2 \) (for the definition see e.g. [3]) and so by Lemma 8 in [3], \( G' \) has exactly two
additional zeros in all of \( \mathbb{C} \). This, together with (i), (ii) and (6), shows that \( f \) has only
real zeros if and only if \( C \) is admissible. The proof of Theorem 1 is now complete.

We would like to thank Hellerstein and Williamson for bringing Lemma 8 in [3] to
our attention. This considerably shortened our original proof.

REFERENCES

4. S. Hellerstein, L. C. Shen and J. Williamson, Reality of the zeros of an entire function and its

DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE, LONDON SW7 2AZ, UNITED KINGDOM

Current address (A. Hinkkanen): Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

Current address (J. Rossi): Department of Mathematics, Virginia Polytechnic Institute, Blacksburg, Virginia 24061