PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 92, Number 1, September 1984

FUNCTIONS OF WELL-BOUNDED OPERATORS

HAROLD E. BENZINGER[1]

ABSTRACT. It is shown that if A is a well-bounded operator of type (B), and if f is of bounded variation and piecewise monotone, then $f(A)$ is also well bounded of type (B).

1. Introduction. Let C be a simple, nonclosed rectifiable arc in the complex plane. For a complex valued function f defined on C, let

$$|||f|||_C = \sup(|f|, C) + \operatorname{var}(f, C),$$

where $\sup(|f|, C)$ is the supremum on C of $|f|$, and $\operatorname{var}(f, C)$ is the total variation of f on C. Let X denote a complex Banach space and let A denote a bounded linear operator on X.

1.1. DEFINITION. The operator A is *well bounded* if there exist an arc C and a constant $K > 0$ such that for all polynomials p, $\|p(A)\| \leqslant K|||p|||_C$.

Well-bounded operators are important in the spectral theory of linear operators with spectral expansions which are only conditionally convergent, e.g. Fourier expansions in L^p spaces, $p \neq 2$. See [**1**] for applications to groups and semigroups of linear operators, and to ordinary differential operators. §2 of [**1**] contains a summary of the theory of well-bounded operators. A systematic development is found in [**2**, Part 5]. The value of the theory arises from the existence of a *functional calculus*: a homomorphism from a Banach algebra of functions on C into the algebra of bounded linear operators on X. Furthermore, there is a modified Riemann-Stieltjes integral on C which gives a representation of the homomorphism. See below, and also [**1**, Propositions 2.1, 2.3]. The functional calculus always exists on the algebra of absolutely continuous functions on C, but for the special class of well-bounded operators of type (B) (see below, and also [**2**, p. 315]), and thus always in the case that X is reflexive, the functional calculus can be extended to $\mathrm{BV}(C)$, the algebra of functions of bounded variation on C, with norm $|||f|||_C$.

In this paper we show that for well-bounded operators of type (B), there is a substantial subset of functions f in $\mathrm{BV}(C)$ such that $f(A)$ is also well bounded.

Since the arc C is the image of a finite, closed real interval under the arc-length parameterization, there is no loss of generality in assuming that C is in fact a real interval (see [**1**, Proposition 2.8]).

Received by the editors September 19, 1983.
1980 *Mathematics Subject Classification*. Primary 47A60.

[1] Part of this work was done while the author was on sabbatical leave at the University of Duisburg and the Technical University of Aachen, West Germany.

©1984 American Mathematical Society
0002-9939/84 $1.00 + $.25 per page

1.2. DEFINITION. A *spectral family* on the interval $J = [a, b]$ is a projection-valued function $E\colon J \to \mathscr{B}(X)$ (the space of bounded linear operators on X) satisfying:

(i) $\sup\{\|E(\lambda)\|\colon \lambda \in J\} = K < \infty$;

(ii) $E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\min(\lambda, \mu))$ for $\lambda, \mu \in J$;

(iii) $E(\lambda)$ is right continuous on J in the strong operator topology;

(iv) $E(\lambda)$ has a left-hand limit in the strong operator topology at each point of J;

(v) $E(b) = I$ (the identity operator on X).

For $f \in \mathrm{BV}(J)$, let $\int_J^r f(\lambda)\, dE(\lambda)$ denote the strong limit of Riemann-Stieltjes sums such that the intermediate point in each interval is the right endpoint. For the proof that this limit exists, see [**2**, Chapter 17]. Let

$$\int_J^{\oplus} f(\lambda)\, dE(\lambda) = f(a)E(a) + \int_J^r f(\lambda)\, dE(\lambda). \tag{1.3}$$

1.4. THEOREM [**2**, CHAPTER 17]. *The bounded linear operator A is a well-bounded operator of type* (B) *on J if and only if there exists a spectral family $E(\lambda)$ on J such that*

$$A = \int_J^{\oplus} \lambda\, dE(\lambda). \tag{1.5}$$

For every $f \in \mathrm{BV}(J)$,

$$f(A) = \int_J^{\oplus} f(\lambda)\, dE(\lambda) \tag{1.6}$$

defines an algebra homomorphism of $\mathrm{BV}(J)$ *into* $\mathscr{B}(X)$, *such that*

$$\|f(A)\| \leqslant K\|\|f\|\|_J. \tag{1.7}$$

1.8. DEFINITION. A real-valued function f on J is *piecewise monotone* if J is the union of finitely many intervals $J_i = [a_{i-1}, a_i]$, $i = 1, \ldots, m$, with $a_0 = a$, $a_m = b$, such that f is monotone on each J_i.

We shall prove the following theorem.

1.9. THEOREM. *If f in* $\mathrm{BV}(J)$ *is piecewise monotone, and if A is a well-bounded operator of type* (B) *on J, then $f(A)$ is a well-bounded operator of type* (B) (*on any compact interval containing* $f(J)$).

1.10. REMARK. For the case f is strictly monotone *and* continuous, see [**1**, Lemma 4.3].

The author wishes to acknowledge a helpful conversation with Professor Earl Berkson.

2. The case that f is monotone. In this section we prove Theorem 1.9 in the case that f is monotone on all of J. Let A be a well-bounded operator of type (B) on J with spectral family $\{E(\lambda)\colon \lambda \in J\}$. Assume f is bounded and monotone nondecreasing on J, with $\alpha = f(a)$, $\beta = f(b)$, $H = [\alpha, \beta]$. For μ in H, define

$$S(\mu) = \{\lambda \in J\colon F(\lambda) \leqslant \mu\}, \qquad \lambda(\mu) = \sup\{\lambda\colon \lambda \in S(\mu)\}. \tag{2.1}$$

Note that

$$S(\mu)=\begin{cases}[a,\lambda(\mu)] & \text{if } f \text{ is left continuous at } \lambda(\mu),\\ [a,\lambda(\mu)) & \text{otherwise.}\end{cases} \tag{2.2}$$

We define a family of projections $F(\mu)$ on H by

$$F(\mu)=\begin{cases}E(\lambda(\mu)) & \text{if } f \text{ is left continuous at } \lambda(\mu),\\ E(\lambda^{-}(\mu)) & \text{otherwise.}\end{cases} \tag{2.3}$$

($E(\lambda_0^-)x = \lim_{\lambda\uparrow\lambda_0} E(\lambda)x$, which exists for all x in X and all λ_0 in J by property (iv) of a spectral family.)

Let $\{\nu_k : k = 1,2,\dots\}$ denote the at most countable set in J where f fails to be left continuous, and define

$$\gamma_k = \lim_{\lambda\uparrow\nu_k} f(\lambda), \qquad \rho_k = f(\nu_k) - \gamma_k > 0, \tag{2.4}$$

$$h(\lambda)=\begin{cases}0, & \lambda \neq \nu_k,\\ \rho_k, & \lambda = \nu_k\end{cases} \qquad g = f - h. \tag{2.5}$$

Then g is left continuous on J. (If ν_k is a cluster point of the set of left discontinuities, we use the fact that $\sum \rho_k < \infty$.)

2.6. Lemma. *$\{F(\mu) : \mu \in H\}$ is a spectral family on H.*

Proof. $F(\mu)$ clearly satisfies properties (i), (ii), (v) of Definition 1.2. Property (iv) holds for F since $\lambda(\mu)$ is a monotone nondecreasing function of μ, and since (iv) holds for E. To establish the right continuity of F, let $\mu_0 \in H$, $x \in X$. Let $\lambda_0 = \lim_{\mu\downarrow\mu_0} \lambda(\mu)$. Then $\lambda_0 = \lambda(\mu_0)$, since otherwise there exists λ, $\lambda(\mu_0) < \lambda < \lambda_0$, such that $f(\lambda) > \mu_0$ but $f(\lambda) \leqslant \mu$ for all $\mu > \mu_0$. Suppose first that f is left continuous at $\lambda_0 = \lambda(\mu_0)$. Then for $\mu > \mu_0$,

$$F(\mu)x - F(\mu_0)x = \begin{cases}E(\lambda(\mu))x - E(\lambda_0)x & \text{if } f \text{ is left continuous at } \lambda(\mu),\\ E(\lambda^{-}(\mu))x - E(\lambda_0)x & \text{otherwise.}\end{cases}$$

In each case, the right continuity of $E(\lambda)$ guarantees that this vector can be made arbitrarily small by selecting $\mu > \mu_0$ sufficiently close to μ_0. Now consider the case that f is not left continuous at λ_0. Then $f(\lambda_0) > \lim_{\lambda\uparrow\lambda_0} f(\lambda) = \gamma_0$, so for $\gamma_0 \leqslant \mu < f(\lambda_0)$, $F(\mu)$ is constant and therefore right continuous.

Let B denote the well-bounded operator of type (B) on H given by

$$B = \int_H^{\oplus} \mu \, dF(\mu). \tag{2.7}$$

We prove that $f(A)$ is well bounded by showing

2.8. Lemma. *$f(A) = B$.*

Proof. First we note that since $\sum \rho_k < \infty$ and $\|E(\lambda)\| \leqslant K$, we have

$$h(A) = \sum_1^{\infty} \rho_k \left[E(\nu_k) - E(\nu_k^-)\right] \tag{2.9}$$

in the strong operator topology. Let $x \in X$ and $\varepsilon > 0$ be given. Let $\{\lambda_k\}_0^n$ be a partition of J containing the points $\nu_1, \ldots, \nu_m$ for some m, such that

$$\left\| g(A)x - \sum_1^n g(\lambda_k)[E(\lambda_k) - E(\lambda_{k-1})]x \right\| < \varepsilon,$$

$$\left\| h(A)x - \sum_1^m \rho_k[E(\nu_k) - E(\nu_k^-)]x \right\| < \varepsilon.$$

Let $\{\mu_j\}_0^q$ denote the partition of H consisting of all points $f(\lambda_k)$ where λ_k is in the above partition, and include also those points γ_i corresponding to $f(\nu_i)$. With respect to this partition,

$$\begin{aligned}
&\sum_1^q \mu_j[F(\mu_j) - F(\mu_{j-1})] \\
&\qquad = \sum_1^m f(\nu_k)[E(\nu_k) - E(\nu_k^-)] + \sum g(\lambda_k)[E(\lambda_k) - E(\lambda_{k-1})] \\
&\qquad = \sum_1^m \rho_k[E(\nu_k) - E(\nu_j^-)] + \sum_1^q g(\lambda_k)[E(\lambda_k) - E(\lambda_{k-1})].
\end{aligned}$$

Refining the $\{\mu_j\}$ partition, if necessary, we have

$$\left\| Bx - \sum \mu_i[F(\mu_j) - F(\mu_{j-1})]x \right\| < \varepsilon.$$

Since this only induces a further refinement of the partition of J, we have $\|Bx - f(A)x\| < 2\varepsilon$.

2.10. LEMMA. *If f: $J \to \mathbf{R}$ is bounded and monotone nonincreasing, then $f(A)$ is well bounded of type* (B).

PROOF. It suffices to show that $-A$ is well bounded of type (B) if A is. Directly from the definition of total variation, we see that $\|p(-A)\| \leqslant K\|p\|_{-J}$, so $-A$ is well bounded. To show that $-A$ is of type (B), it suffices to show that for every x in X, $f \to f(-A)x$ is a compact linear map of $\mathrm{AC}(J)$ into X [**2**, Theorem 17.14, (ii)]. Since this property holds for A, and $-A = g(A)$, with $g(\lambda) = -\lambda$, the result follows.

3. The case that f is piecewise monotone. Let $E(\lambda)$ be a spectral family on J, and assume $J = \bigcup_1^m J_i$, wshere $J_i = [a_{i-1}, a_i]$, $a_0 = a$, $a_m = b$. We define subspaces X_i of X, $i = 0, \ldots, m$:

$$X_0 = \{E(a)x : x \in X\},$$

$$X_i = \{[E(\lambda) - E(a_{i-1})]x : x \in X, \lambda \in J_i\}, \qquad i = 1, \ldots, m.$$

Using the defining properties of a spectral family, it is easy to see that each X_i is closed, and any two have only the zero of X in common. Since any $x \in X$ has the decomposition

$$x = \sum_{i=0}^m x_i = E(a)x + \sum_{i=1}^m [E(a_i) - E(a_{i-1})]x,$$

we have

$$X = X_0 \oplus X_1 \oplus \cdots \oplus X_m. \tag{3.1}$$

In each X_i there is a spectral family $\{E_i(\lambda) : \lambda \in J_i\}$ given by

$$E_0(\lambda) = E(a), \qquad \lambda \in J_0 = \{a\}, \tag{3.2}$$

$$E_i(\lambda) = E(\lambda) - E(a_{i-1}), \qquad \lambda \in J_i. \tag{3.3}$$

Let

$$A = \int_J^{\oplus} \lambda \, dE(\lambda), \qquad A_i = \int_{J_i}^{\oplus} \lambda \, dE_i(\lambda). \tag{3.4}$$

By considering a partition of J including the endpoints a_i, we have

$$A = A_0 \oplus \cdots \oplus A_m, \tag{3.5}$$

and more generally, if $f \in \mathrm{BV}(J)$, $f_i = f|_{J_i}$, we have

$$f(A) = f_0(A_0) \oplus \cdots \oplus f_m(A_m). \tag{3.6}$$

If f is monotone on each J_i, then by the considerations of §2, each $B_i = f_i(A_i)$ is well bounded of type (B). To show that $B = f(A)$ is well bounded, let p be a polynomial on J. Then

$$\begin{aligned}
\|p(B)\| &\leqslant \|p_0(B_0)\| + \cdots + \|p_n(B_m)\| \\
&\leqslant K_0 |||p_0|||_{J_0} + \cdots + K_m |||p_m|||_{J_m} \\
&\leqslant K\left[|p(a)| + \sum_1^m \max(|p|, J_i) + \sum_1^m \mathrm{var}(p, J_i)\right] \\
&\leqslant K |||p|||_J.
\end{aligned}$$

To show that B is of type (B), we again consider the compactness of the mapping of $\mathrm{AC}(J) \to X$ given by $f \to f(B)x$, x fixed. Since the map $f_i \to f_i(B_i)x_i$ is compact, we easily see that the same holds for B.

The spectral family $\{F(\mu) : \mu \in H\}$ of $f(A)$ can be expressed in terms of $E(\lambda)$. For given $\mu \in H$, the level set $S(\mu)$ now consists of the union of finitely many disjoint intervals, which may or may not contain some of their endpoints. Consider the following list giving a correspondence between intervals and differences of projections:

$$\begin{array}{ll}
(\alpha, \beta), & E(\beta^-) - E(\alpha), \\
(\alpha, \beta], & E(\beta) - E(\alpha), \\
[\alpha, \beta) & E(\beta^-) - E(\alpha^-), \\
[\alpha, \beta], & E(\beta) - E(\alpha^-).
\end{array}$$

Then $F(\mu)$ is the sum over each interval in $S(\mu)$, of the corresponding differences of projections.

We now indicate the procedure for replacing J by a simple, nonclosed rectifiable arc C. Let ρ_c denote the arc-length parameterizations of C. Thus there is a real interval J_c whose length is the arc-length of C, such that $\rho_c : J_c \to C, \rho_c \in \mathrm{AC}(J_c)$, and $|\rho_c'| = 1$ a.e. [3, p. 634]. Let $f \in \mathrm{BV}(C)$, and assume $f(C)$ is contained in a simple, nonclosed rectifiable arc Γ, with arc-length parameterization $\rho_\Gamma : J_\Gamma \to \Gamma$. Define $g : J_c \to J_\Gamma$ by

$$g = \rho_\Gamma^{-1} \circ f \circ \rho_c. \tag{3.7}$$

3.8. Definition. f is piecewise monotone on C if g is piecewise monotone on J_c.

3.9. THEOREM. *If A is well bounded of type* (B) *on C, and if $f \in \mathrm{BV}(C)$ maps C into Γ, such that f is piecewise monotone on C, then $f(A)$ is well bounded of type* (B) *on* Γ.

4. An application. Let L be a closed operator with domain $D(L)$ dense in X. Assume zero is in the resolvent set of L, and let $R = (-L)^{-1}$.

4.1. THEOREM. *Assume R is a well-bounded operator of type* (B) *on a simple, nonclosed rectifiable arc C. Let p be a polynomial such that*

(i) *$p(C)$ is contained in a simple, nonclosed rectifiable arc Γ;*

(ii) *$p(-\lambda^{-1}) \neq 0$ for λ in $C - \{0\}$.*

Then $p(L)$ has a well-bounded inverse of type (B).

PROOF. Let $R = \int_C^{\oplus} \lambda \, dE(\lambda)$. If $f \in \mathrm{BV}(C)$ and $\alpha > 0$, let

$$f_\alpha(\lambda) = \begin{cases} f(\lambda), & \rho_c(\lambda, 0) \geqslant \alpha, \\ 0, & \rho_c(\lambda, 0) < \alpha. \end{cases}$$

Then [**1**, Corollary 5.13]

$$Lx = \lim_{\alpha \downarrow 0} \int_C^{\oplus} \left(\frac{-1}{\lambda} \right)_\alpha dE(\lambda)x, \qquad x \in D(L),$$

and for any polynomial p,

$$p(L)x = \lim_{\alpha \downarrow 0} \int_C^{\oplus} p_\alpha\left(\frac{-1}{\lambda} \right) dE(\lambda)x, \qquad x \in D(p(L)).$$

Let $q(\lambda) = \lambda^n p(-1/\lambda)$, where n is the degree of p. Then q is also a polynomial of degree n, $q(\lambda) \neq 0$ for λ in $C - \{0\}$, and $-p^{-1}(-\lambda^{-1}) = -\lambda^n q^{-1}(\lambda)$. By (ii), $-\lambda^n q^{-1}(\lambda) \in \mathrm{BV}(C)$, so

$$S = \int_C^{\oplus} -\lambda^n q^{-1}(\lambda) \, dE(\lambda)$$

is a bounded linear operator. Using the functional calculus,

$$-p(L)Sx = \lim_{\alpha \downarrow 0} \int_C^{\oplus} (1)_\alpha \, dE(\lambda)x = x, \qquad x \in X,$$

and

$$-Sp(L)x = x, \qquad x \in D(p(L)).$$

Thus S is the inverse of $-p(L)$, and S is well bounded of type (B) since the rational function $-\lambda^n q^{-1}(\lambda)$ is piecewise monotone.

REFERENCES

1. H. E. Benzinger, E. Berkson and T. A. Gillespie, *Spectral families of projections, semigroups and differential operators*, Trans. Amer. Math. Soc. **275** (1983), 431–475.

2. H. R. Dowson, *Spectral theory of linear operators*, London Math. Soc. Monographs, No. 12, Academic Press, New York, 1978.

3. J. R. Ringrose, *On well-bounded operators*. II, Proc. London Math. Soc. (3) **13** (1963), 613–638.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801