FUNCTIONS OF WELL-BOUNDED OPERATORS

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ABSTRACT. It is shown that if A is a well-bounded operator of type (B), and if f is of bounded variation and piecewise monotone, then f(A) is also well bounded of type (B).

1. Introduction. Let C be a simple, nonclosed rectifiable arc in the complex plane. For a complex valued function f defined on C, let

\[ \|f\|_C = \sup(|f|, C) + \text{var}(f, C), \]

where \( \sup(|f|, C) \) is the supremum on C of |f|, and \( \text{var}(f, C) \) is the total variation of f on C. Let X denote a complex Banach space and let A denote a bounded linear operator on X.

1.1. Definition. The operator A is well bounded if there exist an arc C and a constant K > 0 such that for all polynomials p, \( \|p(A)\| \leq K \|p\|_C \).

Well-bounded operators are important in the spectral theory of linear operators with spectral expansions which are only conditionally convergent, e.g. Fourier expansions in \( L^p \) spaces, \( p \neq 2 \). See [1] for applications to groups and semigroups of linear operators, and to ordinary differential operators. §2 of [1] contains a summary of the theory of well-bounded operators. A systematic development is found in [2, Part 5]. The value of the theory arises from the existence of a functional calculus: a homomorphism from a Banach algebra of functions on C into the algebra of bounded linear operators on X. Furthermore, there is a modified Riemann-Stieltjes integral on C which gives a representation of the homomorphism. See below, and also [1, Propositions 2.1, 2.3]. The functional calculus always exists on the algebra of absolutely continuous functions on C, but for the special class of well-bounded operators of type (B) (see below, and also [2, p. 315]), and thus always in the case that X is reflexive, the functional calculus can be extended to \( \text{BV}(C) \), the algebra of functions of bounded variation on C, with norm \( \|f\|_C \).

In this paper we show that for well-bounded operators of type (B), there is a substantial subset of functions f in \( \text{BV}(C) \) such that f(A) is also well bounded.

Since the arc C is the image of a finite, closed real interval under the arc-length parameterization, there is no loss of generality in assuming that C is in fact a real interval (see [1, Proposition 2.8]).

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1.2. Definition. A spectral family on the interval \( J = [a, b] \) is a projection-valued function \( E: J \to \mathcal{B}(X) \) (the space of bounded linear operators on \( X \)) satisfying:

(i) \( \sup \{ ||E(\lambda)|| : \lambda \in J \} = K < \infty \);
(ii) \( E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\min(\lambda, \mu)) \) for \( \lambda, \mu \in J \);
(iii) \( E(\lambda) \) is right continuous on \( J \) in the strong operator topology;
(iv) \( E(\lambda) \) has a left-hand limit in the strong operator topology at each point of \( J \);
(v) \( E(b) = I \) (the identity operator on \( X \)).

For \( f \in BV(J) \), let \( \int_J f(\lambda) \, dE(\lambda) \) denote the strong limit of Riemann-Stieltjes sums such that the intermediate point in each interval is the right endpoint. For the proof that this limit exists, see [2, Chapter 17]. Let

\[
\int_J f(\lambda) \, dE(\lambda) = f(a)E(a) + \int_J f(\lambda) \, dE(\lambda).
\]

1.4. Theorem [2, Chapter 17]. The bounded linear operator \( A \) is a well-bounded operator of type (B) on \( J \) if and only if there exists a spectral family \( E(\lambda) \) on \( J \) such that

\[
A = \int_J \lambda \, dE(\lambda).
\]

For every \( f \in BV(J) \),

\[
f(A) = \int_J f(\lambda) \, dE(\lambda)
\]

defines an algebra homomorphism of \( BV(J) \) into \( \mathcal{B}(X) \), such that

\[
||f(A)|| \leq K ||f||_J.
\]

1.8. Definition. A real-valued function \( f \) on \( J \) is piecewise monotone if \( J \) is the union of finitely many intervals \( J_i = [a_{i-1}, a_i], i = 1, \ldots, m \), with \( a_0 = a, a_m = b \), such that \( f \) is monotone on each \( J_i \).

We shall prove the following theorem.

1.9. Theorem. If \( f \) in \( BV(J) \) is piecewise monotone, and if \( A \) is a well-bounded operator of type (B) on \( J \), then \( f(A) \) is a well-bounded operator of type (B) (on any compact interval containing \( f(J) \)).

1.10. Remark. For the case \( f \) is strictly monotone and continuous, see [1, Lemma 4.3].

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2. The case that \( f \) is monotone. In this section we prove Theorem 1.9 in the case that \( f \) is monotone on all of \( J \). Let \( A \) be a well-bounded operator of type (B) on \( J \) with spectral family \( \{ E(\lambda) : \lambda \in J \} \). Assume \( f \) is bounded and monotone nondecreasing on \( J \), with \( \alpha = f(a), \beta = f(b), H = [\alpha, \beta] \). For \( \mu \) in \( H \), define

\[
S(\mu) = \{ \lambda \in J : F(\lambda) \leq \mu \}, \quad \lambda(\mu) = \sup\{ \lambda : \lambda \in S(\mu) \}.
\]
Note that

\[(2.2) \quad S(\mu) = \begin{cases} [a, \lambda(\mu)] & \text{if } f \text{ is left continuous at } \lambda(\mu), \\ [a, \lambda(\mu)) & \text{otherwise.} \end{cases}\]

We define a family of projections \(F(\mu)\) on \(H\) by

\[(2.3) \quad F(\mu) = \begin{cases} E(\lambda(\mu)) & \text{if } f \text{ is left continuous at } \lambda(\mu), \\ E(\lambda(\mu)) & \text{otherwise.} \end{cases}\]

\((E(\lambda_0)x = \lim_{\lambda \uparrow \lambda_0} E(\lambda)x\), which exists for all \(x \in X\) and all \(\lambda_0 \in J\) by property (iv) of a spectral family.\)

Let \(\{v_k: k = 1, 2, \ldots\}\) denote the at most countable set in \(J\) where \(f\) fails to be left continuous, and define

\[(2.4) \quad \gamma_k = \lim_{\lambda \uparrow v_k} f(\lambda), \quad \rho_k = f(v_k) - \gamma_k > 0,\]

\[(2.5) \quad h(\lambda) = \begin{cases} 0, & \lambda \neq v_k, \\ \rho_k, & \lambda = v_k \end{cases} \quad g = f - h.\]

Then \(g\) is left continuous on \(J\). (If \(v_k\) is a cluster point of the set of left discontinuities, we use the fact that \(\sum \rho_k < \infty\).)

2.6. Lemma. \(\{ F(\mu): \mu \in H \}\) is a spectral family on \(H\).

Proof. \(F(\mu)\) clearly satisfies properties (i), (ii), (v) of Definition 1.2. Property (iv) holds for \(F\) since \(\lambda(\mu)\) is a monotone nondecreasing function of \(\mu\), and since (iv) holds for \(E\). To establish the right continuity of \(F\), let \(\mu_0 \in H, x \in X\). Let \(\lambda_0 = \lim_{\mu \uparrow \mu_0} \lambda(\mu)\). Then \(\lambda_0 = \lambda(\mu_0)\), since otherwise there exists \(\lambda, \lambda(\mu_0) < \lambda < \lambda_0\), such that \(f(\lambda) > \mu_0\) but \(f(\lambda) \leq \mu\) for all \(\mu > \mu_0\). Suppose first that \(f\) is left continuous at \(\lambda_0 = \lambda(\mu_0)\). Then for \(\mu > \mu_0\),

\[
F(\mu)x - F(\mu_0)x = \begin{cases} E(\lambda(\mu))x - E(\lambda_0)x & \text{if } f \text{ is left continuous at } \lambda(\mu), \\ E(\lambda(\mu))x - E(\lambda_0)x & \text{otherwise.} \end{cases}
\]

In each case, the right continuity of \(E(\lambda)\) guarantees that this vector can be made arbitrarily small by selecting \(\mu > \mu_0\) sufficiently close to \(\mu_0\). Now consider the case that \(f\) is not left continuous at \(\lambda_0\). Then \(f(\lambda_0) > \lim_{\lambda \uparrow \lambda_0} f(\lambda) = \gamma_0\), so for \(\gamma_0 < \mu < f(\lambda_0), F(\mu)\) is constant and therefore right continuous.

Let \(B\) denote the well-bounded operator of type (B) on \(H\) given by

\[(2.7) \quad B = \int_H^\oplus \mu \, dF(\mu).\]

We prove that \(f(A)\) is well bounded by showing

2.8. Lemma. \(f(A) = B\).

Proof. First we note that since \(\sum \rho_k < \infty\) and \(\|E(\lambda)\| \leq K\), we have

\[
(2.9) \quad h(A) = \sum_{1}^{\infty} \rho_k \left[ E(v_k) - E(v_k^-) \right] \]

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in the strong operator topology. Let \( x \in X \) and \( \varepsilon > 0 \) be given. Let \( \{ \lambda_k \}_0^n \) be a 
partition of \( J \) containing the points \( r_1, \ldots, r_m \) for some \( m \), such that
\[
\left\| g(A)x - \sum_{k=1}^n g(\lambda_k) [E(\lambda_k) - E(\lambda_{k-1})] x \right\| < \varepsilon,
\]
\[
\left\| h(A)x - \sum_{k=1}^m \rho_k [E(\nu_k) - E(\nu_{k-1})] x \right\| < \varepsilon.
\]
Let \( \{ \mu_j \}_0^q \) denote the partition of \( H \) consisting of all points \( f(\lambda_k) \) where \( \lambda_k \) is in the 
above partition, and include also those points \( \gamma_i \) corresponding to \( f(\nu_j) \). With respect 
to this partition,
\[
\sum_{j=1}^q \mu_j \left[ F(\mu_j) - F(\mu_{j-1}) \right]
\]
\[
= \sum_{k=1}^m f(\nu_k) [E(\nu_k) - E(\nu_{k-1})] + \sum_{k=1}^n g(\lambda_k) [E(\lambda_k) - E(\lambda_{k-1})]
\]
\[
= \sum_{k=1}^m \rho_k [E(\nu_k) - E(\nu_{k-1})] + \sum_{k=1}^q g(\lambda_k) [E(\lambda_k) - E(\lambda_{k-1})].
\]
Refining the \( \{ \mu_j \} \) partition, if necessary, we have
\[
\| Bx - \sum_{j=1}^q \mu_j \left[ F(\mu_j) - F(\mu_{j-1}) \right] x \| < \varepsilon.
\]
Since this only induces a further refinement of the partition of \( J \), we have \( \| Bx - f(A)x \| < 2\varepsilon \).

2.10. **Lemma.** If \( f: J \to \mathbb{R} \) is bounded and monotone nonincreasing, then \( f(A) \) is well 
bounded of type (B).

**Proof.** It suffices to show that \( -A \) is well bounded of type (B) if \( A \) is. Directly 
from the definition of total variation, we see that \( \| p(-A) \| \leq K \| p \|_{-J} \), so \( -A \) is well 
bounded. To show that \( -A \) is of type (B), it suffices to show that for every \( x \in X \),
\( f \to f(-A)x \) is a compact linear map of \( AC(J) \) into \( X \) [2, Theorem 17.14, (ii)]. Since 
this property holds for \( A \), and \( -A = g(A) \), with \( g(\lambda) = -\lambda \), the result follows.

3. **The case that \( f \) is piecewise monotone.** Let \( E(\lambda) \) be a spectral family on \( J \), and 
assume \( J = \bigcup_1^m J_i \), where \( J_i = [a_{i-1}, a_i] \), \( a_0 = a \), \( a_m = b \). We define subspaces \( X_i \) of 
\( X \), \( i = 0, \ldots, m \):
\[
X_0 = \{ E(a)x : x \in X \},
\]
\[
X_i = \{ [E(\lambda) - E(a_{i-1})] x : x \in X, \lambda \in J_i \}, \quad i = 1, \ldots, m.
\]
Using the defining properties of a spectral family, it is easy to see that each \( X_i \) is 
closed, and any two have only the zero of \( X \) in common. Since any \( x \in X \) has the 
decomposition
\[
x = \sum_{i=0}^m x_i = E(a)x + \sum_{i=1}^m [E(a_i) - E(a_{i-1})] x,
\]
we have
\[
X = X_0 \oplus X_1 \oplus \cdots \oplus X_m.
\]
In each $X_i$ there is a spectral family $\{E_i(\lambda) : \lambda \in J_i\}$ given by
\begin{align*}
E_0(\lambda) &= E(a), \quad \lambda \in J_0 = \{a\}, \\
E_i(\lambda) &= E(\lambda) - E(a_{i-1}), \quad \lambda \in J_i.
\end{align*}
Let
\begin{equation}
A = \int_{J} \lambda \, dE(\lambda), \quad A_i = \int_{J_i} \lambda \, dE_i(\lambda).
\end{equation}
By considering a partition of $J$ including the endpoints $a_i$, we have
\begin{equation}
A = A_0 \oplus \cdots \oplus A_m,
\end{equation}
and more generally, if $f \in BV(J)$, $f_i = f|_{J_i}$, we have
\begin{equation}
f(A) = f_0(A_0) \oplus \cdots \oplus f_m(A_m).
\end{equation}
If $f$ is monotone on each $J_i$, then by the considerations of §2, each $B_i = f_i(A_i)$ is well bounded of type $(B)$. To show that $B = f(A)$ is well bounded, let $p$ be a polynomial on $J$. Then
\begin{align*}
\|p(B)\| &\leq \|p_0(B_0)\| + \cdots + \|p_m(B_m)\| \\
&\leq K_0\|p_0\|_{J_0} + \cdots + K_m\|p_m\|_{J_m} \\
&\leq K \left[ |p(a)| + \sum_{i=1}^m \max(|p|, J_i) + \sum_{i=1}^m \text{var}(p, J_i) \right] \\
&\leq K \|p\|_{J}.
\end{align*}
To show that $B$ is of type $(B)$, we again consider the compactness of the mapping of $AC(J) \to X$ given by $f \mapsto f(B)x$, $x$ fixed. Since the map $f_i \mapsto f_i(B_i)x_i$ is compact, we easily see that the same holds for $B$.

The spectral family $\{ F(\mu) : \mu \in H \}$ of $f(A)$ can be expressed in terms of $E(\lambda)$. For given $\mu \in H$, the level set $S^{C}(\mu)$ now consists of the union of finitely many disjoint intervals, which may or may not contain some of their endpoints. Consider the following list giving a correspondence between intervals and differences of projections:
\begin{align*}
(\alpha, \beta), & \quad E(\beta^-) - E(\alpha), \\
(\alpha, \beta], & \quad E(\beta) - E(\alpha), \\
[\alpha, \beta), & \quad E(\beta^-) - E(\alpha^-), \\
[\alpha, \beta], & \quad E(\beta) - E(\alpha^-).
\end{align*}
Then $F(\mu)$ is the sum over each interval in $S(\mu)$, of the corresponding differences of projections.

We now indicate the procedure for replacing $J$ by a simple, nonclosed rectifiable arc $C$. Let $\rho_c$ denote the arc-length parameterizations of $C$. Thus there is a real interval $J_c$ whose length is the arc-length of $C$, such that $\rho_c : J_c \to C, \rho_c \in AC(J_c)$, and $|\rho'_{c}| = 1$ a.e. [3, p. 634]. Let $f \in BV(C)$, and assume $f(C)$ is contained in a simple, nonclosed rectifiable arc $\Gamma$, with arc-length parameterization $\rho_{\Gamma} : J_{\Gamma} \to \Gamma$. Define $g : J_c \to J_{\Gamma}$ by
\begin{equation}
g = \rho_{G}^{-1} \circ f \circ \rho_c.
\end{equation}
3.8. **Definition.** $f$ is piecewise monotone on $C$ if $g$ is piecewise monotone on $J_c$. 

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3.9. Theorem. If \( A \) is well bounded of type (B) on \( C \), and if \( f \in \text{BV}(C) \) maps \( C \) into \( \Gamma \), such that \( f \) is piecewise monotone on \( C \), then \( f(A) \) is well bounded of type (B) on \( \Gamma \).

4. An application. Let \( L \) be a closed operator with domain \( D(L) \) dense in \( X \). Assume zero is in the resolvent set of \( L \), and let \( R = (-L)^{-1} \).

4.1. Theorem. Assume \( R \) is a well-bounded operator of type (B) on a simple, nonclosed rectifiable arc \( C \). Let \( p \) be a polynomial such that

(i) \( p(C) \) is contained in a simple, nonclosed rectifiable arc \( \Gamma \);
(ii) \( p(-\lambda^{-1}) \neq 0 \) for \( \lambda \) in \( C - \{0\} \).

Then \( p(L) \) has a well-bounded inverse of type (B).

Proof. Let \( R = \int_C \lambda \ dE(\lambda) \). If \( f \in \text{BV}(C) \) and \( \alpha > 0 \), let

\[
\begin{cases}
  f_\alpha(\lambda), & \rho_c(\lambda,0) \geq \alpha, \\
  0, & \rho_c(\lambda,0) < \alpha.
\end{cases}
\]

Then [1, Corollary 5.13]

\[
Lx = \lim_{\alpha \to 0^+} \int_C \left( -\frac{1}{\lambda} \right)_\alpha dE(\lambda)x, \quad x \in D(L),
\]

and for any polynomial \( p \),

\[
p(L)x = \lim_{\alpha \to 0^+} \int_C p_\alpha \left( -\frac{1}{\lambda} \right) dE(\lambda)x, \quad x \in D(p(L)).
\]

Let \( q(\lambda) = \lambda^n p(-1/\lambda) \), where \( n \) is the degree of \( p \). Then \( q \) is also a polynomial of degree \( n \), \( q(\lambda) \neq 0 \) for \( \lambda \) in \( C - \{0\} \), and \( -p^{-1}(-\lambda^{-1}) = -\lambda^n q^{-1}(\lambda) \). By (ii), \( -\lambda^n q^{-1}(\lambda) \in \text{BV}(C) \), so

\[
S = \int_C -\lambda^n q^{-1}(\lambda) \ dE(\lambda)
\]

is a bounded linear operator. Using the functional calculus,

\[
-p(L)Sx = \lim_{\alpha \to 0^+} \int_C (1)_\alpha dE(\lambda)x = x, \quad x \in X,
\]

and

\[
-Sp(L)x = x, \quad x \in D(p(L)).
\]

Thus \( S \) is the inverse of \( -p(L) \), and \( S \) is well bounded of type (B) since the rational function \( -\lambda^n q^{-1}(\lambda) \) is piecewise monotone.

References


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