A SHORT PROOF OF A DECOMPOSITION THEOREM OF A VON NEUMANN ALGEBRA

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Abstract. Let $M$ be a von Neumann algebra and $S$ and $T$ be commuting $^*$-automorphisms on $M$ satisfying the equation: $S + S^{-1} = T + T^{-1}$. It is proved that $M$ can be decomposed by a central projection $p$ in $M$ such that $S = T$ on $Mp$ and $S = T^{-1}$ on $M(1 - p)$.

Recently the second author proved the following decomposition theorem.

Theorem 1 [3, Theorem 2.6]. Let $S$ and $T$ be commuting $^*$-automorphisms of a von Neumann algebra $M$ satisfying the operator equation

$$S + S^{-1} = T + T^{-1}.$$ 

Then $M$ can be decomposed by a central projection $p$ in $M$ such that $S = T$ on $Mp$ and $S = T^{-1}$ on $M(1 - p)$.

Some applications of this result are discussed by Haagerup and Skau in the geometrical interpretation of the Tomita-Takesaki theory (cf. [1]). A noncommutative version of Theorem 1 (in the case of automorphism groups) has been studied in [4] with its proof depending on Arveson’s theory of spectral subspaces (cf. [5]). The proof of Theorem 1 is rather long and relies on several technical lemmas. The purpose of this note is to give a short proof of this theorem. In fact, we prove a stronger form of this result (Theorem 2) and get the proof of Theorem 1 as an immediate corollary.

Theorem 2. Let $S$ and $T$ be $^*$-automorphisms of a von Neumann algebra $M$ satisfying the operator equation

(i) $$S + T^{-1}S^{-1}T = T + T^{-1}.$$ 

Then $M$ can be decomposed by a central projection $p$ in $M$ such that $S = T$ on $Mp$ and $S = T^{-1}$ on $M(1 - p)$.
Proof. Consider the \( \ast \)-automorphism \( S^{-1}T \) on \( M \). By [2, Theorem 2] there is a central projection \( p \) in \( M \) such that \( M(1 - p) \) is the smallest subalgebra generated by \( R(S^{-1}T - 1) \), the range of \( (S^{-1}T - 1) \), and \( (S^{-1}T - 1)(Mp) = 0 \). Since
\[
(T^{-1} - S)(S^{-1}T - 1) = T^{-1}S^{-1}T - T - T^{-1} + S = 0 \quad \text{(by (i))},
\]
therefore, \( (T^{-1} - S)(M(1 - p)) = 0 \). This completes the proof of the theorem.

Corollary. If \( S \) and \( T \) commute then we get the proof of Theorem 1.

References

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