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CHARACTERIZING THE TOPOLOGY OF INFINITE-DIMENSIONAL σ-COMPACT MANIFOLDS

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ABSTRACT. A metric space (X, d), which is a countable union of finite-dimensional compacta, is a manifold modelled on the space $l_2^f = \{(x_i) \in l_2$: all but finitely many $x_i = 0\}$ iff X is an ANR and the following condition holds: given $\varepsilon > 0$, a pair of finite-dimensional compacta (A, B) and a map $f: A \to X$ such that $f|B$ is an embedding, there is an embedding $g: A \to X$ such that $g|B = f|B$ and $d(f(x), g(x)) < \varepsilon$ for all $x \in A$. An analogous condition characterizes manifolds modelled on the space $\Sigma = \{(x_i) \in l_2: \sum_{i=1}^{\infty}(ix_i)^2 < \infty\}$.

1. Introduction. In this note we will deal with the manifolds modelled on the following pre-Hilbert spaces:

$$l_2^f = \{(x_i) \in l_2\text{: all but finitely many } x_i = 0\}$$

and

$$\Sigma = \left\{(x_i) \in l_2\text{: } \sum_{i=1}^{\infty}(ix_i)^2 < \infty\right\}.$$

The spaces l_2^f and Σ represent the minimal and maximal topological types of infinite-dimensional, σ-compact, locally convex metric linear spaces in the following sense: every infinite-dimensional, σ-compact, locally compact metric linear space contains a topological copy of l_2^f and can be topologically embedded in Σ (see [**4**, p. 274]). Several natural pairs of infinite-dimensional spaces have a structure of (l_2, l_2^2)-manifolds (cf. [**3, 5, 8, 10**]). To recognize them the following characterization was elaborated (cf. [**1, 3, 14, 16, 20**]): the pair (M, N) of metric spaces is an (l_2, l_2^f)-manifold pair iff M is an l_2-manifold, N is the countable union of finite-dimensional compacta and the following condition holds:

(1) given $\varepsilon > 0$, a pair (A, B) of finite-dimensional compacta and a map $f: (A, B) \to (M, N)$ such that $f|B$ is an embedding, there exists an embedding $v: A \to N$ such that $v|B = f|B$ and $d(f(x), v(x)) < \varepsilon$ for all $x \in A$, where d is a metric on M.

This condition can be used to recognize l_2^f-manifolds. But there are situations when we do not know if a given space has a completion homeomorphic to l_2 and therefore (1) cannot be applied (e.g. in the case of an $\aleph_0$-dimensional, nonlocally convex, metric linear space). In [**9**, TC] a question is posed for intrinsic topological characterizations of l_2^f-manifolds and Σ-manifolds without considering suitable completions. In this note we give the following characterizations:

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Let (X, d) be an absolute neighborhood retract which is a countable union of finite-dimensional compacta. Then X is an l_2^f-manifold iff the following condition holds:

(2) given a pair (A, B) of finite-dimensional compacta, a map $f: A \to X$ such that $f|B$ is an embedding, and $\varepsilon > 0$, there is an embedding $v: A \to X$ such that $v = f$ on B and $d(f(x), v(x)) < \varepsilon$ for $x \in A$.

If a σ-compact ANR space Y satisfies the condition (2) for every pair of compacta, then it is a Σ-manifold.

These results are obtained, analogously as in [**18** and **19**], by considering the projections p_X: $X \times l_2^f \to X$ and p_Y: $Y \times \Sigma \to Y$, respectively, and are based on a theorem of Toruńczyk [**17**] stating that $X \times l_2^f$ is an l_2^f-manifold for σ-finite-dimensional and σ-compact ANR space X, and $Y \times \Sigma$ is a Σ-manifold for σ-compact ANR space Y. Since the spaces under consideration are incomplete we cannot use Bing's shrinking criterion applied in [**18, 19**]. Instead of Bing's shrinking criterion we use some lemma concerning a stabilizing sequence of maps (see [**13**, Lemma 4] and §2 here).

To unify the proofs for the l_2^f- and Σ-case an abstract scheme of approximation of maps by homeomorphisms is given in §3. The characterizations of l_2^f- and Σ-manifolds are formulated in §4.

Applying our characterization, it is not hard to prove that every $\aleph_0$-dimensional linear metric space (i.e. having a Hamel basis of cardinality $\aleph_0$) is homeomorphic to l_2^f. This fact and other consequences of our criteria are given in [**6**].

2. Preliminaries. In this section we fix notation and formulate some facts needed later.

Suppose that X and Y are topological spaces. We write $\operatorname{cov}(X)$ for family of all open covers of X and $C(X, Y)$ for the space of all continuous functions from X to Y topologized by the "limitation topology" in which each $f \in C(X, Y)$ has the collection $\{V(f, \mathcal{U})\colon \mathcal{U} \in \operatorname{cov}(Y)\}$ as a basis of neighborhoods, where $V(f, \mathcal{U}) = \{g \in C(X, Y)\colon$ for each $x \in X$ there exists $U \in \mathcal{U}$ containing both $f(x)$ and $g(x)\}$. Members of $V(f, \mathcal{U})$ are said to be $\mathcal{U}$-close to f.

Suppose that Y is a metric space and d is a metric on Y. For $\alpha \in C(Y, (0, 1))$ and $f \in C(X, y)$, let

$$V(f, \alpha) = \{g \in C(X, Y)\colon\ d(f(x), g(x)) < \alpha(f(x)) \text{ for each } x \in X\}.$$

The members of $V(f, \alpha)$ are said to be α-close to f. A map $h: X \times [0, 1] \to Y$ is said to be an α-homotopy if, for each $x \in X$, $\operatorname{diam}(h(\{x\} \times [0, 1])) < \alpha(h(x, 0))$.

The following facts are known (cf. Theorem 4.1 of [**4**] and [**13**], [**15**]):

(A) For every $\mathcal{U} \in \operatorname{cov}(Y)$ there exists $\alpha \in C(Y, (0, 1))$ such that for every $f \in C(X, Y)$ $V(f, \mathcal{U}) \supset V(f, \alpha)$.

(B) For every $\mathcal{U} \in \operatorname{cov}(Y)$ there exists a metric ρ on Y compatible with d such that the cover of Y by open balls of radius 1 (with respect to ρ) is a refinement of $\mathcal{U}$.

The map $f \in C(X, Y)$ is a *near-homeomorphism* if for every $\mathcal{U} \in \operatorname{cov}(Y)$ there exists a homeomorphism of X onto Y which is $\mathcal{U}$-close to f.

A closed subset A of X is called a *Z-set* in X ($A \in Z(X)$) if $\{f \in C(Q, X)\colon f(Q) \cap A = \emptyset\}$ is dense in $C(Q, X)$, where Q denotes the Hilbert cube.

(C) $A \in Z(X)$ iff for every n the set $\{f \in C(I^n, X): f(I^n) \cap A = \emptyset\}$ is dense in $C(I^n, X)$.

We shall need the following fact [**13**, Lemma 4]:

(D) Let (Y, d) be a metric space and $\{Y_n\}_{n=1}^{\infty}$ be a closed increasing cover of Y. For $n = 1, 2, \ldots$, let g_n: $X \to Y$ be a surjective map from a metric space X satisfying the following conditions:

(i) $g_n | g_n^{-1}(Y_n)$: $g_n^{-1}(Y_n) \to Y_n$ is one-to-one, and for every $y \in Y_n$ and every neighborhood V of $g_n^{-1}(y)$ in X, there exists an open neighborhood $\mathcal{U}$ of y in Y with $g_n^{-1}(\mathcal{U}) \subset V$;

(ii) $g_{n+1} | g_n^{-1}(Y_n) = g_n | g_n^{-1}(Y_n)$;

(iii) $g_{n+1} | X \setminus g_n^{-1}(Y_n)$ is α_n-close to $g_n | X \setminus g_n^{-1}(Y_n)$, where

$$\alpha_n(y) = 2^{-n}\{\min 1, d(y, Y_n)\},$$

with $\alpha_0(y) = 1$ for all y.

Then the map g, defined on the subset $Z = \bigcup_{n=1}^{\infty} g_n^{-1}(Y_n)$ by $g(z) = \lim g_n(z)$, is a homeomorphism of Z onto Y such that $d(g(x), g_1(x)) < 1$ for $x \in Z$.

Let $f \in C(X, Y)$, and let A be a closed subset of Y. The space $(X, f)_A$ is defined to be the set $(X \setminus f^{-1}(A)) \cup A$ with the topology generated by open subsets of $X \setminus f^{-1}(A)$ and by sets of the form $f^{-1}(U \setminus A) \cup (U \cap A)$, where U is open in Y. We define the map $f_A: X \to (X, f)_A$ by the formula

$$f_A(x) = \begin{cases} x & \text{for } x \in X \setminus f^{-1}(A), \\ f(x) & \text{for } x \in f^{-1}(A). \end{cases}$$

It is an easy consequence of the definition that the function p_A: $(X, f)_A \to Y$, defined by $p_A f_A = f$, is continuous and satisfies the following condition:

(E) for every $y \in A$ and every neighborhood V of $p_A^{-1}(y)$ in $(X, f)_A$ there exists an open neighborhood U of y in Y with $p_A^{-1}(U) \subset V$.

Let us observe that $(Y \times Z, \pi)_A$, where π: $Y \times Z \to Y$ is the projection, is a cartesian product of Y and Z reduced over A (denoted by $(Y \times Z)_A$), see [**4**, p. 25]. If X and Y are metrizable, then for every closed subset A of Y $(X, f)_A$ is metrizable as a subset of $(Y \times X)_A$.

3. The strong universality property for compacta. A metric ANR space is said to be *strongly universal for (finite-dimensional) compacta* if, for each map $f: A \to X$ of a (finite-dimensional) compactum, each closed subset B of A such that $f|B$ is an embedding, and each $\varepsilon > 0$, there exists an embedding $g: A \to X$ such that g is ε-close to f and $g|B = f|A$. The space l_2^f is strongly universal for finite-dimensional compacta and the space Σ is strongly universal for compacta.

1. LEMMA. *Let X be a metric ANR space which is strongly universal for (finite-dimensional) compacta, let $f: A \to X$ be a map of a compactum and let B be a closed subset of A such that $f|B$ is an embedding, $f(A \setminus B) \subset X \setminus f(B)$ (and $A \setminus B$ is a countable union of finite-dimensional compacta). Then given $\mathcal{U} \in \operatorname{cov}(X \setminus f(B))$ there exists an embedding $g: A \to X$ such that $g|B = f|B$ and $g|A \setminus B$ is $\mathcal{U}$-close to $f|A \setminus B$.*

PROOF. Fix a metric d on X. By A there exists a continuous function

$$\beta\colon X \setminus f(B) \to (0, 1)$$

such that every two β-close maps into $X \setminus f(B)$ are $\mathcal{U}$-close. Let α: $A \to [0,1)$ be a continuous function such that $\alpha^{-1}(0) = B$ and $\alpha(a) < \beta(f(a))$ for all $a \in A \setminus B$. Let $A \setminus B = \bigcup_{n=1}^{\infty} A_n$, where $\{A_n\}_{n=1}^{\infty}$ is a increasing sequence of finite-dimensional compacta. We let $\varepsilon_n = \inf\{\alpha(a) : a \in A_n\}$. Then $\{\varepsilon_n\}$ is a decreasing sequence of positive numbers with $\lim \varepsilon_n = 0$. We shall inductively construct a sequence of maps $\{f_n : A \to X\}$ such that:

(a) $f_n(A \setminus B) \subset X \setminus f(B)$ and $f_n|B = f|B$,
(b) $f_n|A_{n-1} = f_{n-1}|A_{n-1}$,
(c) $f_n|A_n \cup B$ is an embedding,
(d) $d(f_n(a), f_{n-1}(a)) \leq 2^{-n}\alpha(a)$ for all a.

We let $f_0 = f$. Assume that f_{n-1} has been already constructed. Because $X \setminus f(B)$ is an ANR the restriction $h \mapsto h|A_n$ is an open map from $C(A \setminus B, X \setminus f(B))$ to $C(A_n, X \setminus f(B))$ (see [**19**, Lemma 1.3]). Hence, using strong universality property, we can find an embedding $v_n : A_n \to X \setminus f(B)$ such that $v_n|A_{n-1} = f_{n-1}|A_{n-1}$ and v_n is so close to $f_{n-1}|A_n$ that there is an extension $g_n : A \setminus B \to X \setminus f(B)$ of v_n which is $2^{-n}\varepsilon_n$-homotopic to $f_{n-1}|A \setminus B$. Let $h_n : (A \setminus B) \times [0,1] \to X \setminus f(B)$ be a $2^{-n}\varepsilon_n$-homotopy with $h_n(a,0) = f_{n-1}(a)$ and $h_n(a,1) = g_n(a)$ for $a \in A \setminus B$. Let v_n be a compact neighborhood of A_n in $A \setminus B$ such that

$$\operatorname{diam}(h_n(\{a\} \times [0,1])) < 2^{-n}\alpha(a) \quad \text{for } a \in V_n.$$

Then the map f_n defined by

$$f_n(a) = \begin{cases} h_n(a, \lambda_n(a)) & \text{for } a \in A \setminus B, \\ f(a) & \text{for } a \in B, \end{cases}$$

where $\lambda_n : A \to [0,1]$ is such that $\lambda_n^{-1}(0) \supset A \setminus V_n$ and $\lambda_n^{-1}(1) = A_n$, has the required properties. Since $A \setminus B = \bigcup_{n=1}^{\infty} A_n$, the map $g = \lim f_n$ is an embedding of A onto X such that $g|B = f|B$. By (d), for each $a \in A \setminus B$

$$d(f(a), g(a)) \leq \sum_{h=1}^{\infty} 2^{-n}\alpha(a) = \alpha(a) < \beta(f(a)).$$

Thus $g|A \setminus B$ is $\mathcal{U}$-close to $f|A \setminus B$.

2. LEMMA. *Let X be a metric ANR space. If X is strongly universal for (finite-dimensional) compacta, then every (finite-dimensional) compact subset of X is a Z-set in X.*

PROOF. Fix a metrix d on X. Let X be strongly universal for finite-dimensional compacta and let K be a finite-dimensional, compact subset of X. Let $f : I^n \to X$ be a map of an n-dimensional cube into X, and let $\varepsilon > 0$ be given. We shall construct a map $g : I^n \to X$ which is ε-close to f and such that $g(I^n) \cap K = \emptyset$. By strong universality of X there is an embedding $v : I^n \to X$ which is $\frac{1}{2}\varepsilon$-close to f. We can regard the set $B = v(I^n) \cup K$ as a subset of $I^m \times \{0\} \subset I^m \times [0,1]$, for some $m \geq n$. Then the inclusion i: $B \to X$ can be extended to a map h: $A \to X$, where A is a compact neighborhood of B in $I^m \times [0,1]$. By strong universality of X there is an embedding w: $A \to X$ such that $w|B = i$. By compactness of $v(I^n)$ there is $t \in (0,1]$ such that $d(w(x,t), w(x,0)) < \frac{1}{2}\varepsilon$ for $x \in v(I^n)$. Let $g(y) = w(v(y),t)$ for $y \in I^n$. Then g is the required map.

Analogously we can prove that every compact subset of a strongly universal for compacta, ANR space X is a Z-set in X.

3. LEMMA. *Let X be an ANR which is strongly universal for finite-dimensional compacta. Then every compact subset of X which is a countable union of finite-dimensional compacta is a Z-set in X.*

PROOF. Let Y be a complete metric space which contains X and satisfies the following condition:

(3) for every compact subset A in X, $A \in Z(X)$ iff $A \in Z(Y)$ (see [**15**, Proposition 4.1]).

Let K be a compact subset of X. By Lemma 2 K is a countable union of Z-sets in X. By (3) K is a countable union of Z-sets in Y. Because Y is complete and K is closed in Y, $K \in Z(Y)$ (see [**4**, p. 151]). By (3) again $K \in Z(X)$.

4. Near-homeomorphisms between σ-compacta. A metric ANR space X has the *estimated extension property for compacta* if, for each open subset G of X, and each $\mathcal{U} \in \operatorname{cov}(G)$, and each homeomorphism $v: A \to B$ between compacta in G such that v is $\mathcal{U}$-homotopic to id_A, there exists a space homeomorphism $h: X \to X$ extending v, and such that h is $\operatorname{st}\mathcal{U}$-close to id_X.

4. THEOREM. *Let X and Y be metric ANR spaces which are countable unions of (finite-dimensional) compacta. Suppose that X has the estimated extension property for compacta and Y is strongly universal for (finite-dimensional) compacta. Let f: $X \to Y$ be a map with the property that, for every compactum A in Y and closed subset B of A, the map $f_A|X \setminus f^{-1}(B)$: $X \setminus f^{-1}(B) \to (X, f)_A \setminus B$ is a near-homeomorphism. Then f is a near-homeomorphism.*

PROOF. We will only consider the case when X and Y are countable unions of finite-dimensional compacta, and Y is strongly universal for finite-dimensional compacta. Let $X = \bigcup_{n=1}^{\infty} A_n$ and $Y = \bigcup_{n=1}^{\infty} B_n$, where A_n and B_n are finite-dimensional compacta for $n = 1, 2, \ldots$. Let d be any metric on Y. By (B), it is enough to check that there is a homeomorphism h of X onto Y such that $d(h(x), f(x)) < 1$ for $x \in X$. We shall inductively construct a sequence $\{C_n\}_{n=0}^{\infty}$ of compact subsets of Y and a sequence $\{h_n\}_{n=0}^{\infty}$ of homeomorphisms of X onto $(X, f)_{C_n} = X_n$ such that, for $n = 1, 2, \ldots$:

$(\mathrm{a})_n$ $C_n \supset B_n \cup C_{n-1}$;

$(\mathrm{b})_n$ $h_n(A_n) \subset C_n$;

$(\mathrm{c})_n$ $h_n|h_{n-1}^{-1}(C_{n-1}) = h_{n-1}|h_{n-1}^{-1}(C_{n-1})$;

$(\mathrm{d})_n$ $p_n h_n|X \setminus h_{n-1}^{-1}(C_{n-1})$ is α_n-close to $p_{n-1}h_{n-1}|X \setminus h_{n-1}^{-1}(C_{n-1})$, where $\alpha_n: Y \setminus C_{n-1} \to (0,1)$ is defined by $\alpha_n(y) = 2^{-n}\min\{1, d(y, C_{n-1})\}$ and $p_n: X_n \to Y$ is the map defined by $p_n f_{C_n} = f$.

We let $C_0 = \emptyset$ and $h_0 = \mathrm{id}$. Assume that h_i: $X \to X_i$ and X_i satisfying $(\mathrm{a})_i$, $(\mathrm{b})_i$, $(\mathrm{c})_i$ and $(\mathrm{d})_i$ for $0 \le i \le n$ have been constructed. Note that $p_n(X_n \setminus C_n) \subset Y \setminus C_n$. let $\mathcal{U}$ be an open cover of $Y \setminus C_n$ such that

$$V(p_n|X_n \setminus C_n, \operatorname{st}^3 \mathcal{U}) \subset V(p_n|X_n \setminus C_n, \alpha_{n+1}).$$

By strong universality of Y and Lemma 1, there is an embedding v of $D_{n+1} = h_n(A_{n+1}) \cup C_n \subset X_n$ into Y such that $v|C_n = \mathrm{id}_{C_n}$ and $v|D_{n+1} \setminus C_n$ is $\mathcal{U}$-homotopic to $p_n|D_{n+1} \setminus C_n$. Take

$$C_{n+1} = B_{n+1} \cup v(D_{n+1}) \cup H((D_{n+1} \setminus C_n) \times [0,1]),$$

where $H: (D_{n+1} \setminus C_n) \times [0,1] \to Y$ is a $\mathcal{U}$-homotopy with $H(x,0) = p_n(x)$ and $H(x,1) = v(x)$ for $x \in D_{n+1} \setminus C_n$. Because $f_{C_n}|X \setminus f^{-1}(C_n)$ is a homeomorphism of $X \setminus f^{-1}(C_n)$ onto $X_n \setminus C_n$, by the assumption about the map f, there exists a homeomorphism $g_{n+1}: X_n \to X_{n+1}$ such that $g_{n+1}|C_n = \mathrm{id}$ and $g_{n+1}|X_n \setminus C_n$ is $p_{n+1}^{-1}(\mathcal{U})$-homotopic to the map $f_{(C_{n+1},C_n)}|X_n \setminus C_n$, where $f_{(C_{n+1},C_n)}$ is defined by the equality $f_{(C_{n+1},C_n)} \circ f_{C_n} = f_{C_{n+1}}$. The embeddings $g_{n+1}|D_{n+1}$ and $(p_{n+1}C_{n+1})^{-1}v$ are $\mathrm{st}(p_{n+1}^{-1}(\mathcal{U}))$-homotopic. Since X_{n+1}, being homeomorphic to X, has the estimated extension property for compacta there exists a homeomorphism u_{n+1} of X_{n+1} onto itself which is $\mathrm{st}^2(p_{n+1}^{-1}(\mathcal{U}))$-close to the identity and such that $u_{n+1}q_{n+1}D_{n+1} = (p_{n+1}C_{n+1})^{-1}v$. We let $h_{n+1} = u_{n+1}g_{n+1}h_n$. Then $p_{n+1}h_{n+1}$ is $\mathrm{st}^2(\mathcal{U})$-close to $p_{n+1}g_{n+1}h_n$ and hence $p_{n+1}h_{n+1}$ is $\mathrm{st}^3(\mathcal{U})$-close to p_nh_n. It is easy to see that $(\mathrm{a})_{n+1}$, $(\mathrm{b})_{n+1}$, $(\mathrm{c})_{n+1}$ and $(\mathrm{d})_{n+1}$ are satisfied.

Since each p_n satisfies (E) we apply (D) to the sequences $\{C_n\}$ and $\{p_nh_n\}$. Therefore the map $h = \lim p_nh_n$ is a homeomorphism of X onto Y such that $d(f(x), h(x)) < 1$ for all $x \in X$.

5. Characterization of l_2^f- and Σ-manifolds.

5. THEOREM. *Let X be an ANR space which is a countable union of finite-dimensional compacta. Then X is an l_2^f-manifold iff it is strongly universal for finite-dimensional compacta.*

PROOF. By a theorem of Torunczyk [**17**], $X \times l_2^f$ is an l_2^f-manifold and therefore has the estimated extension property for compacta. Given an open set $U \subset X$ and a compact set A in X the space $(U \times l_2^f)_{A\cap U}$ in an ANR (see [**13**, Lemma 5]). We will prove that $A \cap U$ is a Z-set in $(U \times l_2^f)_{A\cap U}$. Take $g: I^n \to (U \times l_2^f)_{A\cap U}$ and $\varepsilon > 0$. Let $\pi_U: U \times l_2^f \to (U \times l_2^f)_A$ denote the projection. Given $\varepsilon > 0$ there exists a map $q: (U \times l_2^f)_A \times U \times l_2^f$ such that $\pi_U q$ is $\varepsilon/2$-close to the identity (see [**13**, Lemma 5]). By Lemma 3 A is a Z-set in X, hence $A \cap U$ is a Z-set in U and $(A \cap U) \times l_2^f$ is a Z-set in $U \times l_2^f$ (see [**4**, p. 151]). Thus there exists a map $f: I^n \to U \times l_2^f$ such that $f(I^n) \cap ((A \cap U) \times l_2^f) = \emptyset$ and so close to qg that $\pi_U f$ is $\varepsilon/2$-close to $\pi_U qg$. Hence $\pi_U f$ is ε-close to g and $\pi_U f(I^n) \cap (A \cap U) = \emptyset$. It means that $A \cap U$ is a Z-set in $(U \times l_2^f)_{A\cap U}$. By [**15**] the projection $\pi_U: (U \times l_2^f) \to (U \times l_2^f)_{A\cap U}$ is a near-homeomorphism. Thus the projection $\pi: X \times l_2^f \to X$ satisfies the assumption of Theorem 4. Hence π is a near-homeomorphism and X is an l_2^f-manifold.

6. THEOREM. *Let X be an ANR. Then X is a Σ-manifold iff it is σ-compact and is strongly universal for compacta.*

PROOF. We can repeat the proof of Theorem 5 replacing finite-dimensional compacta by compacta and l_2^f by Σ.

6. Questions. Let us formulate questions which are closely related to the problem of identifying σ-compact manifolds.

Let G be a locally contractible, metrizable topological group which is a countable union of finite-dimensional compacta and is not locally compact. We do not know whether G must be strongly universal for finite-dimensional compacta, and therefore an l_2^f-manifold (see [**9**, TCG]).

Henderson and Walsh in [**11**] have constructed an example of a cell-like decomposition $\mathcal{G}$ of l_2^f such that the decomposition space $l_2^f/\mathcal{G}$ is not homeomorphic to l_2^f but $l_2^f/\mathcal{G} \times [0,1]$ is homeomorphic to l_2^f. Let us mention that there is no such decomposition of the Hilbert space l_2 (see [**12**]). It means that the space l_2^f behaves more like finite-dimensional euclidean spaces than like the Hilbert space l_2. Hence the following question is interesting.

7. QUESTION. Let $\mathcal{G}$ be a cell-like decomposition of l_2^f such that $l_2^f/\mathcal{G}$ is a countable union of finite-dimensional compacta. Is $l_2^f/\mathcal{G} \times [0,1]$ (or $l_2^f/\mathcal{G} \times [0,1]^2$) homeomorphic to l_2^f?

Because the condition (2) is difficult to verify for some spaces it would be useful to find some new conditions characterizing l_2^f-manifolds. Examples of Henderson and Walsh [**11**] show that the following conditions are not sufficient to assure than ANR X, which is a countable union of finite-dimensional compacta, is an l_2^f-manifold:

(4) every compact subset of X is a Z-set in X,

(5) every map $f: \oplus_{n=1}^{\infty} I^n \to X$ of the countable, free union of finite-dimensional cubes is strongly approximable by maps $g: \oplus_{n=1}^{\infty} I^n \to X$, for which the collection $\{g(I^n)\}_{n=1}^{\infty}$ is discrete.

Note that there is a topologically complete separable metric AR space, which is not homeomorphic to l_2, but which satisfies (4) (see [**2**]).

Let ε be the class of dense linear subspaces of l_2 which are countable unions of compacta with defined transfinite dimension. For every $E \in \varepsilon$ let $\gamma(E)$ be the infimum of ordinals α such that E is a countable union of compacta with transfinite dimension $< \alpha$ (see [**4**, p. 282]).

8. QUESTION. Let $E_1, E_2 \in \varepsilon$ and let $\gamma(E_1) = \gamma(E_2)$. Are E_1 and E_2 homeomorphic?

9. QUESTION. Is it true that for every $\alpha \in \{\gamma(E): E \in \varepsilon\}$ there is $E_\alpha \in \varepsilon$, with $\gamma(E) = \alpha$ and which is topologically universal for all compacta with transfinite dimension $< \alpha$?

Note that every σ-compact linear subspace E of l_2 which is universal for compacta is homeomorphic to Σ (see [**7**]).

ADDED IN PROOF. The proofs of Theorems 5 and 6 are based on the following theorem of Toruńczyk [**15**, Proposition 5.1]:

(6) if A is a compact Z-set in a σ-compact (σ-finite-dimensional compact) ANR X, then the projection $\pi_A: X \times E \to (X \times E)_A$ is a near homeomorphism, where $E = \Sigma$ or l_2^f.

It has been observed recently that the above theorem is false. Moreover it turns out that the strong universality properties do not characterize l_2^f- and Σ-manifolds among σ-compact ANR's. Theorems 5 and 6 are true if, in addition, the space X satisfies the following condition:

(7) every compact subset A of X is a strong Z-set in X (i.e. given an open cover $\mathcal{U}$ of X there exists $f: X \to X$, $\mathcal{U}$-close to the identity map, such that $f(X) \cap V = \varnothing$ for some neighborhood V of A).

Proofs are the same and use (6) which holds if A is a strong Z-set in X. Let us note that for σ-compact ANR's (7) is equivalent to (5). Details of proofs and related examples will appear in [**21**].

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