REGULARITY OF THE DISTANCE FUNCTION
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ABSTRACT. A coordinate-free proof is given of the fact that the distance function \( \delta \) for a \( C^k \) submanifold \( M \) of \( \mathbb{R}^n \) is \( C^k \) near \( M \) when \( k \geq 2 \). The result holds also when \( k = 1 \) if \( M \) has a neighborhood with the unique nearest point property. The differentiability of \( \delta \) in the \( C^1 \) case is seen to follow directly from geometric considerations.

In the study of analysis and geometry, the function that measures the distance to a submanifold plays an important role. Let \( M \) be a submanifold of \( \mathbb{R}^n \), and let \( \delta: \mathbb{R}^n \to \mathbb{R} \) be the distance function for \( M \), \( \delta(x) = \text{dist}(x, M) \). If \( M \) is \( C^k \), then \( \delta \) is easily seen to be \( C^{k-1} \) near \( M \), since \( \delta \) is always continuous and can be written in terms of the directions normal to \( M \). It is the case, however, that \( \delta \) is actually \( C^k \) near \( M \) when \( k \geq 2 \), and even when \( k = 1 \) under certain circumstances. As Krantz and Parks [4] point out, this fact deserves to be better known that it is.

The regularity of \( \delta \) was first considered in [1], and the proof for the case \( k \geq 2 \) is found in [2]. The combined results (including the case \( k = 1 \)) are given in [4] in a proof based on the work in [1].

The purpose of this note is to present a simple, coordinate-free proof of the following theorem and its \( C^1 \) analog.

**THEOREM 1.** Let \( M \subset \mathbb{R}^n \) be a compact, \( C^k \) submanifold with \( k \geq 2 \). Then \( M \) has a neighborhood \( U \) so that \( \delta \) is \( C^k \) on \( U - M \).

In the \( C^1 \) case, the additional hypothesis is needed that some neighborhood of \( M \) have the unique nearest point property. (See [1, 4].) A neighborhood \( U \) of \( M \) has this property if for every \( x \in U \) there is a unique point \( P(x) \in M \) so that \( \delta(x) = \text{dist}(x, P(x)) \). The map \( P: U \to M \) is called the projection onto \( M \).

**LEMMA.** Let \( M \) satisfy the hypothesis of Theorem 1. Then \( M \) has a neighborhood \( U \) with the unique nearest point property, and the projection map \( P: U \to M \) is \( C^{k-1} \).

**PROOF.** This is just the tubular neighborhood theorem with the added observation that the projection \( P \) factors through the map that creates the neighborhood. (See [3].)

Let

\[ \nu(M) = \{(p, v) \in \mathbb{R}^n \times \mathbb{R}^n | p \in M \text{ and } v \perp T_p M\} \]

be the normal bundle for \( M \); it is a \( C^{k-1} \) manifold of dimension \( n \). Define the \( C^{k-1} \) map \( F: \nu(M) \to \mathbb{R}^n \) by \( F(p, v) = p + v \). The Jacobian \( F_* \) is easily seen to be nonsingular along the zero section \( \{(p, 0) \in \nu(M)\} \). By the inverse function theorem and the compactness of \( M \), there is an \( \varepsilon > 0 \) such that \( F \) restricted to
\[ \nu_\varepsilon(M) = \{(p, v) \in \nu(M): |v| < \varepsilon\} \text{ is a } C^{k-1} \text{ diffeomorphism onto a neighborhood } U \text{ of } M. \text{ On } U \text{ the map } P \text{ is the composition } \\
U \xrightarrow{F^{-1}} \nu_\varepsilon(M) \rightarrow M, \]

where the last map is projection onto the first factor. Q.E.D.

In [4] M is said to have positive reach of at least \( \varepsilon \). The largest possible neighborhood \( U \) on which \( P \) is well defined is determined in part by the local extrinsic geometry of \( M \) inside \( \mathbb{R}^n \): the extrinsic curvature of \( M \) governs the location of the singularities of the map \( F: \nu(M) \rightarrow \mathbb{R}^n \). (See [5, §6].)

Proof of Theorem 1. On the neighborhood \( U \) where \( P \) is well defined, the distance function is given by \( \delta(x) = \|x - P(x)\| \). For \( v \in \mathbb{R}^n \), let \( D_v \) denote differentiation in the direction \( v \). Then for \( x \in U - M, \)

\[
(D_v \delta^2)(x) = 2(x - P(x)) \cdot (v - D_v P(x)) = 2(x - P(x)) \cdot v,
\]

since \( D_v P(x) \) is tangent to \( M \). Hence

\[
(*) \quad (\text{grad } \delta^2)(x) = 2(x - P(x)),
\]

which is \( C^{k-1} \), and so \( \delta \) is \( C^k \) on \( U - M \). Q.E.D.

In the \( C^1 \) case one needs to examine the behavior of the difference quotient.

Theorem 2. Let \( M \) be \( C^1 \) and suppose \( U \) is a neighborhood of \( M \) with the unique nearest point property. Then \( \delta \) is \( C^1 \) on \( U - M \).

Proof. A simple argument (see [1, 4.8(4)]) shows that \( P: U \rightarrow M \) is continuous. Thus, it suffices to show that (*) holds on \( U - M \). If this is not the case, then there is some point \( x \in U - M \) and some vector \( v \in \mathbb{R}^n \) such that

\[
(1) \quad \lim_{t \to 0^+} \frac{\delta^2(x + tv) - \delta^2(x)}{t} < 2(x - P(x)) \cdot v
\]

or

\[
(2) \quad \lim_{t \to 0^+} \frac{\delta^2(x + tv) - \delta^2(x)}{t} > 2(x - P(x)) \cdot v.
\]

In the first case, one can find a fixed \( \varepsilon > 0 \) and then choose \( t > 0 \) arbitrarily close to zero such that

\[
\delta^2(x + tv) < \delta^2(x) + 2(x - P(x)) \cdot tv - t\varepsilon.
\]

It follows, then, that

\[
\text{dist}^2(x, P(x + tv)) = \|(x + tv - P(x + tv)) - tv\|^2 \\
= \delta^2(x + tv) - 2(x + tv - P(x + tv)) \cdot tv + t^2\|v\|^2 \\
< \delta^2(x) + 2(P(x + tv) - P(x)) \cdot tv - t\varepsilon - t^2\|v\|^2.
\]

By the continuity of \( P \), \( t \) can be chosen small enough so that

\[
\text{dist}(x, P(x + tv)) < \delta(x) = \text{dist}(x, P(x)).
\]

Then \( x \) is closer to \( P(x + tv) \) than to \( P(x) \), a contradiction.

Similarly, (2) leads to \( \text{dist}(x + tv, P(x)) < \text{dist}(x + tv, P(x + tv)) \). The theorem follows. Q.E.D.
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REMARKS. (1) With some modifications, the same proofs will work when $M$ is a submanifold of a Riemannian manifold.

(2) If $M$ is a hypersurface of the form $M = \{ x \in \mathbb{R}^n \mid \rho(x) = 0 \}$, where $\rho$ is a $C^k$ function with $d\rho \neq 0$ on $M$, then one can form the signed distance function

$$\bar{\delta}(x) = \begin{cases} 
\delta(x) & \text{for } \rho(x) \geq 0, \\
-\delta(x) & \text{for } \rho(x) \leq 0.
\end{cases}$$

It is easy to see that $\bar{\delta}$ is $C^k$ on all of $U$.

(3) In the $C^1$ case, the regularity of $M$ does not enter into the proof. $M$ can be replaced by any closed set in $\mathbb{R}^n$, and $U$ by any open set on which the projection $P: U \to M$ is well defined. (For the original treatment of this, see [1].)

(4) The extra hypothesis in the $C^1$ case is essential. The distance function for the curve $y = |x|^{3/2}$ in $\mathbb{R}^2$ is not differentiable at any point on the $y$-axis. See [4] for details.

For further remarks and examples, the reader is directed to the references, especially [4].

REFERENCES