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G-PROJECTIVE GROUPS

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ABSTRACT. Let TF be the category of torsion free abelian groups of finite rank and homomorphisms. For G in TF let PC(G) be the projective class in TF generated by $\{G\}$. THEOREM. PC(G) *consists exactly of groups of the form* $P \oplus F$, *where* F *is finite rank free and* P *is* G*-projective* ($P \oplus P' \cong G^n$ *for some positive integer* n).

Let TF be the category of torsion free abelian groups of finite rank and homomorphisms. An object G in TF will be called simply a "group". In [**A–L**], Arnold and Lady studied *G-projective groups*: direct summands of direct sums of finitely many copies of G. In this note we give another characterization of G-projective groups.

Define PC(G), the projective class in TF generated by G, to be the class of all groups projective with respect to all exact sequences in TF with respect to which G is projective. The study of projective classes was proposed by Fuchs [**F**, Problem 46] and has been undertaken by various authors [**H, R-W-W, S, V-W 1–3, Wa, W**]. Our main theorem is that for any G in TF, PC(G) consists precisely of the groups of the form $P \oplus F$, where F is a finite rank free group and P is G-projective.

LEMMA 1 [**V-W 3**, COROLLARY 1.2]. *Given groups G and H, there is a rank-2 group C with Q an epimorphic image of C such that $Q\operatorname{Hom}(H, C \otimes G) \cong Q\operatorname{Hom}(H, (Z \oplus Z) \otimes G)$. Moreover, the isomorphism is induced by any embedding, $0 \to Z \oplus Z \to C$.*

DEFINITION. Let A, H be groups and f: $A \to H$. The map f is called G-balanced if the induced map f^*: $\operatorname{Hom}(G, A) \to \operatorname{Hom}(G, H)$ is epic.

LEMMA 2. *Let H in* PC(G) *have no free summands. Then there is a group B such that H is an epimorphic image of B but* $\operatorname{Hom}(H, B) = 0$.

PROOF. Let F be a full free subgroup of H. By Lemma 1 there is a rank-2 group C with an epimorphism q: $C \to Q$ such that

$$Q\operatorname{Hom}(H, C \otimes F) \cong Q\operatorname{Hom}(H, F \oplus F) = 0.$$

Define e: $C \otimes F \to QH$ by $e(c \otimes f) = q(c)f$. Then e is onto. Set $B = e^{-1}(H) \subseteq C \otimes F$. Note that $\operatorname{Hom}(H, B) = 0$ since $\operatorname{Hom}(H, C \otimes F) = 0$.

LEMMA 3. *Let H in* PC(G) *have no free summands. Suppose A is a group and f in* $\operatorname{Hom}(A, H)$ *is G-balanced. Then f is a splitting epimorphism.*

PROOF. Construct B and e as in Lemma 2. The epimorphism $f \oplus e$: $A \oplus B \to H$ is G-balanced since f is. Hence there is a splitting map h: $H \to A \oplus B$. But $h(H) \subseteq A$ because $\operatorname{Hom}(H, B) = 0$. Thus h is a splitting map for f.

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COROLLARY 4. *Let H in* PC(G) *have no free summand. Then* $\operatorname{Hom}(G,H)G = H$.

PROOF. Let $A = \operatorname{Hom}(G,H) \otimes G$ and $f\colon A \to H$ be given by $g \otimes x \to g(x)$. For h in $\operatorname{Hom}(G,H)$, define h' in $\operatorname{Hom}(G,A)$ by $h'(x) = h \otimes x$. Then $f^*(h') = h$. The previous lemma applies to show that f is a splitting epimorphism. In particular, $\operatorname{Hom}(G,H)G = H$.

PROPOSITION 5. *Let H in* PC(G) *have no free summand. Then the following are equivalent*:
(a) *H is G-projective.*
(b) *H is a quasi-summand of G^n for some n.*
(c) $\operatorname{Hom}(G,H)$ *is finitely generated as a right $E(G)$-module.*

PROOF. (a)$\to$(b) is obvious. For (b)$\to$(c) first note that $\operatorname{Hom}(G,H)$ is finitely generated over $E = E(G)$ if and only if $\operatorname{Hom}(G^n,H)$ is finitely generated over E. Let M be the ring of $n \times n$ matrices over E. Then $\operatorname{Hom}(G^n,H)$ is a right module over M in the natural way and, since M is finitely generated as a right E module, it suffices to prove that $\operatorname{Hom}(G^n,H)$ is finitely generated over M. We can assume, for some positive integer t and group X, that $G^n > H \oplus X > tG^n$. Thus, if π is projection of $H \oplus X$ onto H, then $\pi t \in \operatorname{Hom}(G^n,H)$ and $\operatorname{Hom}(G^n,H) > (\pi t)M > t\operatorname{Hom}(G^n,H)$. Since $\operatorname{Hom}(G^n H)/t\operatorname{Hom}(G^n,H)$ is finite, the result follows.

To show (c)$\to$(a), let $\{f_1,\dots,f_n\}$ be a set of generators for $\operatorname{Hom}(G,H)$ as a right E-module. Define $f\colon G^n \to H$ by $f(x_1,\dots,x_n) = f_1(x_1) + \cdots + f_n(x_n)$. Then f is an epimorphism in view of Corollary 4. Moreover, given $g\colon G \to H$, there exist $e_1,\dots,e_n$ in E such that $g = f_1e_1 + \cdots + f_ne_n$. Define $g'\colon G \to G^n$ by $g'(x) = (e_1(x),\dots,e_n(x))$. Then $fg' = g$ and the epimorphism $f\colon G^n \to H$ is G-balanced. Thus, f splits and H is G-projective.

THEOREM 6. *Let G, H be torsion free abelian groups of finite rank. Then H belongs to* PC(G) *if and only if $H = P \oplus F$, where P is G-projective and F is a finite rank free group.*

PROOF. It follows easily from the definition that any group of the form $P \oplus F$ is in PC(G). To show the converse, it suffices to prove that if H in PC(G) has no free summand, then H is G-projective.

Using Lemma 1, choose a rank-2 group C and an epimorphism $q\colon C \to Q$ such that $Q\operatorname{Hom}(H, C \otimes G) = Q\operatorname{Hom}(H, G \oplus G)$. Next choose a maximal Z-independent set $\{f_1,\dots,f_n\}$ in $\operatorname{Hom}(G,H)$ and define $f\colon (C \otimes G)^n \to QH$ by $f(c_1 \otimes x_1,\dots,c_n \otimes x_n) = q(c_1)f_1(x_1) + \cdots + q(c_n)f_n(x_n)$. The map f is onto by Corollary 4 (note that every element of $Q\operatorname{Hom}(G,H)$ is of the form $q(c_1)f_1 + \cdots + q(c_n)f_n$). Let $A = f^{-1}(H)$. We will show the epimorphism $f\colon A \to H$ is G-balanced. Let g belong to $\operatorname{Hom}(G,H)$. Then $g = q(c_1)f_1 + \cdots + q(c_n)f_n$ for some choice of $c_1,\dots,c_n$ in C. For x in G, $a = (c_1 \otimes x,\dots,c_n \otimes x)$ is in A since $f(a) = q(c_1)f_1(x) + \cdots + q(c_n)f_n(x) = g(x)$ is in H. If we define $g'\colon G \to A$ by $g'(x) = (c_1 \otimes x,\dots,c_n \otimes x)$, then $fg' = g$. This implies that f is G-balanced.

Since H is in PC(G), there is a splitting map $e\colon H \to A \subseteq (C \otimes G)^n$. But $Q\operatorname{Hom}[H,(C \otimes G)]^n \cong [Q\operatorname{Hom}(H, G \oplus G)]^n$. Therefore, there is a positive integer m such that me can be regarded as an element of $\operatorname{Hom}[H,(G \oplus G)]^n$. It follows that H is a quasi-summand of $(G \oplus G)^n$. The theorem follows from Proposition 5.

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