CREMONA TRANSFORMATIONS
THAT ARE AFFINE AUTOMORPHISMS

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ABSTRACT. We present the condition on which a Cremona transformation induces an automorphism of $\mathbb{A}^2$.

1. In this paper the ground field is assumed to be an algebraically closed field of characteristic zero. Let $C$ be an irreducible algebraic curve in $\mathbb{P}^2$. Then let us call $C$ a curve of type I if $C - \{P\} \cong \mathbb{A}^1$ for some point $P \in C$, and a curve of type II if $C \setminus L \cong \mathbb{A}^1$ for some line $L$ [3]. Let $B$ and $G$ be the groups of birational transformations of $\mathbb{P}^2$ (i.e., Cremona transformations) and automorphisms of $\mathbb{A}^2$, respectively. Identifying $\mathbb{P}^2 - L$ with $\mathbb{A}^2$, where $L$ is a line, we regard $G$ as a subgroup of $B$ hereafter. Let $F(f)$ denote the set of fundamental points of $f \in B$. Then let us call $L$ a general line if $L \cap F(f) = \emptyset$. Two elements $f_1$ and $f_2$ of $B$ are said to be equivalent if there is a projective transformation $p$ satisfying $f_2 = pf_1$. If $f$ is equivalent to an element of $G$, then $f(L)$ is a curve of type II for any general line $L$. On the other hand, in case $C \setminus L \cong \mathbb{A}^1$, there is an automorphism of $\mathbb{P}^2 - L$ by which $G$ is carried to a line [1]. Thus there is a close relation between automorphisms of $\mathbb{A}^2$ and curves of type II. We shall give a formulation of this fact in Remark 1. Here we present the condition on which a Cremona transformation induces an automorphism of $\mathbb{A}^2$.

THEOREM. Let $f$ be a Cremona transformation. Suppose the exceptional curve of $f$ is irreducible and $f(L)$ is a curve of type II for a general line $L$. Then $f$ is equivalent to an element of $G$, hence $f(L')$ is also a curve of type II for every general line $L'$.

Note that the theorem is not necessarily true in case we drop the condition of irreducibility (see Example 2). On the other hand, in case $f(L)$ is a curve of type I, $f(L')$ is not necessarily a curve of the type for another general line $L'$ (see Example 3).

2. Proof of the Theorem. For a curve $C$ of type I, we define $R$ as in [3], i.e., let $(e_1, \ldots, e_t)$ be the sequence of the multiplicities of the infinitely near singular points of $P$ with degree $d \geq 3$. Then $R = R(C)$ is $d^2 - \sum_{i=1}^{t} e_i^2 - e_t + 1$. We have shown that $R(C) \geq 2$ if $C$ is of type II [3]. Let $D$ and $D'$ be the exceptional curves of $f$ and $f^{-1}$ respectively. Let $\sigma_i$, $i = 1, \ldots, r$, be blow-ups such that $f \cdot \sigma$ is a morphism, where $\sigma = \sigma_1 \cdots \sigma_r$. Since $D$ is irreducible, so is $D'$. Hence the centers of the blow-ups are unique if the number $r$ is minimal. Thus $\sigma^{-1}[D]$ is first contracted by $f \cdot \sigma$, where $[\ ]$ denotes the proper transform. Note that $\sigma^{-1}[D]$ is a nonsingular rational curve with the self-intersection number $-1$ by Castelnuovo's
criterion for contracting a curve [2]. Since $L$ is a general line, the transform $\sigma^{-1}(L)$ has self-intersection number 1. The transforms $\sigma^{-1}(L)$ and $\sigma^{-1}[D]$ meet at only one point $Q$, since $f(L)$ is of type I. Therefore the intersection number of those curves at $Q$ is $e$, which is the degree of $D$. Suppose $e \geq 2$. Then, after the contraction of $\sigma^{-1}[D]$, the image of $L$ has a singular point with multiplicity $e$. Hence the degree of $f(L)$ is at least 3. From the above consideration we infer that $R(f(L)) = 2 - e$. This means $f(L)$ is not of type II. Thus we have a contradiction, hence $D$ must be a line. Since $f$ induces an isomorphism between $\mathbb{P}^2 - D$ and $\mathbb{P}^2 - D'$, the curve $D'$ is also a line. Hence $f$ is equivalent to an element of $G$.

3. Let $(X, Y, Z)$ be a set of homogeneous coordinates on $\mathbb{P}^2$, and let $X$, $Y$ and $Z$ denote the lines defined by $X = 0$, $Y = 0$ and $Z = 0$, respectively. Moreover, let $P = (1, 0, 0)$.

REMARK 1. Let $S$ be the set consisting of curves $C$ such that $C \cap Z = \{P\}$ and $C - \{P\} \cong \mathbb{A}^1$. Identifying $\mathbb{P}^2 - Z$ with $\mathbb{A}^2$, we put

$G_1 = \{f \in G|F(f^{-1}) = \{P\} \text{ or } f(P) = P \text{ according to } F(f) \neq 0 \text{ or } F(f) = 0\}$

and

$G_2 = \{f \in G|f[Y] = Y\}$. If $f \in G_2$ and $F(f) \neq 0$, then $F(f^{-1}) = \{P\}$, otherwise the degree of $f^{-1}[Y]$ will be greater than 1. Thus we see that $G_2 \subset G_1$ and $G_2$ becomes a group; hence we consider the set $G_1/G_2$ of equivalence classes of $G_1$ with respect to $G_2$. Then there is a bijection between $G_1/G_2$ and $S$. In fact, let an element $f \in G_1$ correspond to $f[Y]$, which belongs to $S$. Thanks to [1] this mapping is surjective, hence defines the bijection. Thus, we may say that nonsingular rational curves in $\mathbb{A}^2$ passing through the infinite point $P$ can be “parametrized” by $G_1/G_2$.

EXAMPLE 2. Let $g$ be a Cremona transformation defined by $g(X, Y, Z) = (g_1, g_2, g_3)$, where $n \geq 2$ and

$g_1 = XY^{n-1}Z + Z^{n+1}$, \quad $g_2 = Y^{n+1}$, \quad $g_3 = Y^nZ$.

Then $F(g) = \{P\}$ and the exceptional curves of $g$ are $Y$ and $Z$, but lines not passing through $P$ are carried to curves of type II.

EXAMPLE 3. Let $\Delta$ be the curve defined by $XZ^{n-1} = Y^n$, where $n \geq 3$, and $g$ a nonlinear automorphism of $\mathbb{P}^2 - \Delta$. Then $F(g) = \{P\}$ and $g(X)$ is of type I, whereas $f(L)$ is not of type I for any other general line $L$. For details see [3].

REFERENCES


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