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CREMONA TRANSFORMATIONS THAT ARE AFFINE AUTOMORPHISMS

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ABSTRACT. We present the condition on which a Cremona transformation induces an automorphism of $\mathbf{A}^2$.

1. In this paper the ground field is assumed to be an algebraically closed field of characteristic zero. Let C be an irreducible algebraic curve in $\mathbf{P}^2$. Then let us call C a curve of type I if $C - \{P\} \cong \mathbf{A}^1$ for some point $P \in C$, and a curve of type II if $C \backslash L \cong \mathbf{A}^1$ for some line L [**3**]. Let B and G be the groups of birational transformations of $\mathbf{P}^2$ (i.e., Cremona transformations) and automorphisms of $\mathbf{A}^2$, respectively. Identifying $\mathbf{P}^2 - L$ with $\mathbf{A}^2$, where L is a line, we regard G as a subgroup of B hereafter. Let $F(f)$ denote the set of fundamental points of $f \in B$. Then let us call L a general line if $L \cap F(f) = \emptyset$. Two elements f_1 and f_2 of B are said to be equivalent if there is a projective transformation p satisfying $f_2 = pf_1$. If f is equivalent to an element of G, then $f(L)$ is a curve of type II for any general line L. On the other hand, in case $C \backslash L \cong \mathbf{A}^1$, there is an automorphism of $\mathbf{P}^2 - L$ by which C is carried to a line [**1**]. Thus there is a close relation between automorphisms of $\mathbf{A}^2$ and curves of type II. We shall give a formulation of this fact in Remark 1. Here we present the condition on which a Cremona transformation induces an automorphism of $\mathbf{A}^2$.

THEOREM. *Let f be a Cremona transformation. Suppose the exceptional curve of f is irreducible and $f(L)$ is a curve of type* II *for a general line L. Then f is equivalent to an element of G, hence $f(L')$ is also a curve of type* II *for every general line L'.*

Note that the theorem is not necessarily true in case we drop the condition of irreducibility (see Example 2). On the other hand, in case $f(L)$ is a curve of type I, $f(L')$ is not necessarily a curve of the type for another general line L' (see Example 3).

2. Proof of the Theorem. For a curve C of type I, we define R as in [**3**], i.e., let $(e_1, \ldots, e_t)$ be the sequence of the multiplicities of the infinitely near singular points of P with degree $d \geq 3$. Then $R = R(C)$ is $d^2 - \sum_{i=1}^t e_i^2 - e_t + 1$. We have shown that $R(C) \geq 2$ if C is of type II [**3**]. Let D and D' be the exceptional curves of f and f^{-1} respectively. Let σ_i, $i = 1, \ldots, r$, be blow-ups such that $f \cdot \sigma$ is a morphism, where $\sigma = \sigma_1 \cdots \sigma_r$. Since D is irreducible, so is D'. Hence the centers of the blow-ups are unique if the number r is minimal. Thus $\sigma^{-1}[D]$ is first contracted by $f \cdot \sigma$, where [ ] denotes the proper transform. Note that $\sigma^{-1}[D]$ is a nonsingular rational curve with the self-intersection number -1 by Castelnuovo's

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criterion for contracting a curve [**2**]. Since L is a general line, the transfrom $\sigma^{-1}(L)$ has self-intersection number 1. The transforms $\sigma^{-1}(L)$ and $\sigma^{-1}[D]$ meet at only one point Q, since $f(L)$ is of type I. Therefore the intersection number of those curves at Q is e, which is the degree of D. Suppose $e \geq 2$. Then, after the contraction of $\sigma^{-1}[D]$, the image of L has a singular point with multiplicity e. Hence the degree of $f(L)$ is at least 3. From the above consideration we infer that $R(f(L)) = 2 - e$. This means $f(L)$ is not of type II. Thus we have a contradiction, hence D must be a line. Since f induces an isomorphism between $\mathbf{P}^2 - D$ and $\mathbf{P}^2 - D'$, the curve D' is also a line. Hence f is equivalent to an element of G.

3. Let (X, Y, Z) be a set of homogeneous coordinates on $\mathbf{P}^2$, and let X, Y and Z denote the lines defined by $X = 0$, $Y = 0$ and $Z = 0$, respectively. Moreover, let $P = (1, 0, 0)$.

REMARK 1. Let S be the set consisting of curves C such that $C \cap Z = \{P\}$ and $C - \{P\} \cong \mathbf{A}^1$. Identifying $\mathbf{P}^2 - Z$ with $\mathbf{A}^2$, we put

$$G_1 = \{f \in G | F(f^{-1}) = \{P\} \text{ or } f(P) = P \text{ according to } F(f) \neq \emptyset \text{ or } F(f) = \emptyset\}$$

and

$$G_2 = \{f \in G | f[Y] = Y\}.$$

If $f \in G_2$ and $F(f) \neq \emptyset$, then $F(f^{-1}) = \{P\}$, otherwise the degree of $f^{-1}[Y]$ will be greater than 1. Thus we see that $G_2 \subset G_1$ and G_2 becomes a group; hence we consider the set G_1/G_2 of equivalence classes of G_1 with respect to G_2. Then there is a bijection between G_1/G_2 and S. In fact, let an element $f \in G_1$ correspond to $f[Y]$, which belongs to S. Thanks to [**1**] this mapping is surjective, hence defines the bijection. Thus, we may say that nonsingular rational curves in $\mathbf{A}^2$ passing through the infinite point P can be "parametrized" by G_1/G_2.

EXAMPLE 2. Let g be a Cremona transformation defined by $g(X, Y, Z) = (g_1, g_2, g_3)$, where $n \geq 2$ and

$$g_1 = XY^{n-1}Z + Z^{n+1}, \quad g_2 = Y^{n+1}, \quad g_3 = Y^n Z.$$

Then $F(g) = \{P\}$ and the exceptional curves of g are Y and Z, but lines not passing through P are carried to curves of type II.

EXAMPLE 3. Let Δ be the curve defined by $XZ^{n-1} = Y^n$, where $n \geq 3$, and g a nonlinear automorphism of $\mathbf{P}^2 - \Delta$. Then $F(g) = \{P\}$ and $g(X)$ is of type I, whereas $f(L)$ is not of type I for any other general line L. For details see [**3**].

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