KRULL VERSUS GLOBAL DIMENSION
IN NOETHERIAN P.I. RINGS

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Abstract. The Krull dimension of any noetherian P.I. ring is bounded above by its
global (homological) dimension (when finite).

1. Introduction. A longstanding open problem is the conjecture that for a noetherian
ring $R$ of finite global dimension, the Krull dimension of $R$ is no larger than the
global dimension. This conjecture was verified for semiprime noetherian P.I. rings by
Resco, Small, and Stafford [7, Theorem 3.2], and more recently for certain fully
bounded noetherian rings by Brown and Warfield [2, Corollary 12], as follows.

Theorem A. (Brown-Warfield). Let $R$ be a fully bounded noetherian ring
containing an uncountable set $F$ of central units such that the difference of any two
distinct elements of $F$ is a unit. If gl.dim.$(R)$ is finite, then K.dim.$(R) \leq$ gl.dim.$(R)$.

We proceed by applying Theorem A to Laurent series rings, using the following
observation.

Proposition B. If $R$ is a nonzero noetherian P.I. ring, then the Laurent series ring
$R((x))$ is a fully bounded noetherian ring containing an uncountable set $F$ of central
units such that the difference of any two distinct elements of $F$ is a unit.

Proof. Set $T = R((x))$; it is well known that $T$ is noetherian (or see Proposition
2).

By [6, Theorem II.4.1], $R$ satisfies a multilinear identity $f$ with coefficients $\pm 1$.
Then $f$ is satisfied in $R[x]$, and hence in $R[x]/x^nR[x]$, for all positive integers $n$. As
$R[[x]]$ is an inverse limit of the rings $R[x]/x^nR[x]$, it satisfies $f$, whence $T$, being a
central localization of $R[[x]]$, satisfies $f$. Thus $T$ is a P.I. ring. By [1, Theorem 7 or 6,
Theorem II.5.3], $T$ is fully bounded.

Let $F$ be the set of those Laurent series in $T$ with all coefficients either 0 or 1, and
note that $F$ is an uncountable subset of the center of $T$. Any element of $F$, or any
difference of two distinct elements of $F$, has leading coefficient $\pm 1$ and so is a unit
in $T$. □
To obtain our general result requires the following change of rings theorem, proved in §2.

**THEOREM C.** If $R$ is a right noetherian ring and $T$ is the Laurent series ring $R((x))$, then $r.K.dim.(R) = r.K.dim.(T)$ and $r.gl.dim.(R) = r.gl.dim.(T)$. \( \square \)

Our main result is now an immediate consequence of Theorems A and C and Proposition B.

**THEOREM D.** If $R$ is any noetherian P.I. ring for which $gl.dim.(R)$ is finite, then $K.dim.(R) < gl.dim.(R)$. \( \square \)

2. Laurent series rings. Here we consider the Krull and global dimensions of Laurent series rings over arbitrary noetherian rings (not necessarily P.I.).

**DEFINITION.** Let $T = R((x))$ be the Laurent series ring over a ring $R$. Any nonzero element $t \in T$ has the form

\[
t = \sum_{i=0}^{\infty} t_i x^i,
\]

where $n \in \mathbb{Z}$, each $t_i \in R$ and $t_n \neq 0$. The integer $n$ is called the order of $t$, and the element $t_n$ is called the leading coefficient of $t$, which we shall denote by $\lambda(t)$. By convention, $\lambda(0) = 0$. For a right ideal $I$ of $T$, define $\lambda(I) = \{\lambda(t) | t \in I\}$, and observe that $\lambda(I)$ is a right ideal of $R$.

**LEMMA 1.** Let $R$ be a ring, let $T = R((x))$ and let $I, J$ be right ideals of $T$ such that $\lambda(I) = \lambda(J)$ and $\lambda(I)$ is finitely generated, then $I = J$.

**PROOF.** We may assume that $I \neq 0$. Choose nonzero elements $a_1, \ldots, a_m$ in $I$ such that $\lambda(a_1), \ldots, \lambda(a_m)$ generate $\lambda(I)$. After multiplying the $a_i$ by suitable powers of $x$, we may assume that the $a_i$ all have order 0.

Now consider any nonzero element $b \in J$. In showing that $b \in I$, there is no harm in multiplying $b$ by a power of $x$. Hence, we may assume that $b$ has order 0. We construct elements $s_{ij} \in R$ (for $i = 1, \ldots, m$ and $j = 0, 1, 2, \ldots$) such that for each $n = 0, 1, 2, \ldots$, the element

\[
b - \sum_{i=1}^{m} \sum_{j=0}^{n} a_is_{ij}x^j
\]

has order greater than $n$.

Since $\lambda(b) \in \lambda(J) = \lambda(I) = \sum \lambda(a_i)R$, there exist elements $s_{i0} \in R$ such that

\[
\lambda(b) = \lambda(a_1)s_{10} + \cdots + \lambda(a_m)s_{m0}.
\]

As $a_1, \ldots, a_m, b$ all have order 0, the element

\[
b - (a_1s_{10} + \cdots + a_ms_{m0})
\]

must have order greater than 0.

Now assume that we have constructed $s_{ij} \in R$ for $i = 1, \ldots, m$ and $j = 0, 1, \ldots, n$ such that the element

\[
c = b - \sum_{i=1}^{m} \sum_{j=0}^{n} a_is_{ij}x^j
\]
has order greater than $n$. Note that $c \in J$, and let $c_{n+1}$ denote the coefficient of $x^{n+1}$ in $c$. Either $c_{n+1} = 0$ or $c_{n+1} = \lambda(c)$, whence $c_{n+1} \in \lambda(J)$ in either case. There exist elements $s_{i,n+1} \in R$ such that

$$c_{n+1} = \lambda(a_1)s_{1,n+1} + \cdots + \lambda(a_m)s_{m,n+1},$$

and the element

$$c - (a_1s_{1,n+1}x^{n+1} + \cdots + a_ms_{m,n+1}x^{n+1})$$

has order greater than $n + 1$. This completes the induction step.

Finally, setting $d_i = \sum_{j=0}^{\infty} s_{i,j}x^j$ for each $i = 1, \ldots, m$, we conclude that $b = a_1d_1 + \cdots + a_md_m$. Therefore $b \in I$. \qed

**Proposition 2.** Let $R$ be a right noetherian ring, and let $T = R((x))$. Then $T$ is a right noetherian ring and $\text{r.K.dim.}(T) = \text{r.K.dim.}(R)$.

**Proof.** We have a map $\lambda$ from the lattice of right ideals of $T$ to the lattice of right ideals of $R$, and Lemma 1 shows that $\lambda$ preserves strict inclusions. Consequently, $T$ is right noetherian, and $\text{r.K.dim.}(T) \leq \text{r.K.dim.}(R)$.

For each right ideal $I$ of $R$, let $I((x))$ denote the right ideal of $T$ consisting of those elements of $T$ with all coefficients lying in $I$. The map $I \mapsto I((x))$ defines an embedding of the lattice of right ideals of $R$ into the lattice of right ideals of $T$, whence $\text{r.K.dim.}(R) \leq \text{r.K.dim.}(T)$. \qed

**Theorem 3.** Let $R$ be a right noetherian ring, and let $T = R((x))$. Then $\text{r.gl.dim.}(T) = \text{r.gl.dim.}(R)$.

**Proof.** Since $R$ is right coherent, all direct products of flat left $R$-modules are flat [3, Theorem 2.1]. Hence, for each integer $n$, the set $T_n$ consisting of those elements of $T$ with order at least $n$ is a flat left $R$-module. Thus $T$, being the union of the $T_n$, is flat as a left $R$-module. In addition, $R$ is an $(R, T)$-bimodule direct summand of $T$. Therefore $\text{r.gl.dim.}(R) \leq \text{r.gl.dim.}(T)$, by [5, Lemma 1].

We may now assume that $\text{r.gl.dim.}(R) = n < \infty$. Set $S = R[[x]]$. Since $x$ is a central regular element in the Jacobson radical of $S$ and $S/xS \cong R$, [4, Part III, Theorem 10] shows that $\text{r.gl.dim.}(S) = n + 1$. On the other hand, $\text{r.gl.dim.}(T) \leq \text{r.gl.dim.}(S)$ because $T$ is a central localization of $S$. Thus $\text{r.gl.dim.}(T)$ equals either $n$ or $n + 1$.

If $\text{r.gl.dim.}(T) = n + 1$, there exists a right ideal $I$ in $T$ such that $T/I$ has projective dimension $n + 1$. Set $J = I \cap S$ and $A = S/J$, and observe that $A \otimes_S T \cong T/I$. Now $A$ is a finitely generated right $S$-module on which $x$ is a non-zero-divisor, and we observe that $A$ must have projective dimension $n + 1$. According to [4, Part III, Theorem 9], the right $R$-module $A/\text{rad}A$ must have projective dimension $n + 1$, which is impossible.

Therefore $\text{r.gl.dim.}(T) = n$. \qed

**References**


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