PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 92, Number 2, October 1984

KRULL VERSUS GLOBAL DIMENSION IN NOETHERIAN P.I. RINGS

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ABSTRACT. The Krull dimension of any noetherian P.I. ring is bounded above by its global (homological) dimension (when finite).

1. Introduction. A longstanding open problem is the conjecture that for a noetherian ring R of finite global dimension, the Krull dimension of R is no larger than the global dimension. This conjecture was verified for semiprime noetherian P.I. rings by Resco, Small, and Stafford [**7**, Theorem 3.2], and more recently for certain fully bounded noetherian rings by Brown and Warfield [**2**, Corollary 12], as follows.

THEOREM A. (BROWN-WARFIELD). *Let R be a fully bounded noetherian ring containing an uncountable set F of central units such that the difference of any two distinct elements of F is a unit. If* gl.dim.(R) *is finite, then* K.dim.$(R) \leqslant$ gl.dim.(R). □

We proceed by applying Theorem A to Laurent series rings, using the following observation.

PROPOSITION B. *If R is a nonzero noetherian P.I. ring, then the Laurent series ring $R((x))$ is a fully bounded noetherian ring containing an uncountable set F of central units such that the difference of any two distinct elements of F is a unit.*

PROOF. Set $T = R((x))$; it is well known that T is noetherian (or see Proposition 2).

By [**6**, Theorem II.4.1], R satisfies a multilinear identity f with coefficients ± 1. Then f is satisfied in $R[x]$, and hence in $R[x]/x^nR[x]$, for all positive integers n. As $R[[x]]$ is an inverse limit of the rings $R[x]/x^nR[x]$, it satisfies f, whence T, being a central localization of $R[[x]]$, satisfies f. Thus T is a P.I. ring. By [**1**, Theorem 7 or **6**, Theorem II.5.3], T is fully bounded.

Let F be the set of those Laurent series in T with all coefficients either 0 or 1, and note that F is an uncountable subset of the center of T. Any element of F, or any difference of two distinct elements of F, has leading coefficient ± 1 and so is a unit in T. □

Received by the editors October 24, 1983.
1980 *Mathematics Subject Classification*. Primary 16A33, 16A38, 16A55, 16A60.
Key words and phrases. Noetherian ring, polynomial identity, Krull dimension, homological dimension.
[1]The research of both authors was partially supported by grants from the National Science Foundation.

To obtain our general result requires the following change of rings theorem, proved in §2.

THEOREM C. *If R is a right noetherian ring and T is the Laurent series ring $R((x))$, then* r.K.dim.(R) = r.K.dim.(T) *and* r.gl.dim.(R) = r.gl.dim.(T). □

Our main result is now an immediate consequence of Theorems A and C and Proposition B.

THEOREM D. *If R is any noetherian P.I. ring for which* gl.dim.(R) *is finite, then* K.dim.$(R) \leqslant$ gl.dim.(R). □

2. Laurent series rings. Here we consider the Krull and global dimensions of Laurent series rings over arbitrary noetherian rings (not necessarily P.I.).

DEFINITION. Let $T = R((x))$ be the Laurent series ring over a ring R. Any nonzero element $t \in T$ has the form

$$t = \sum_{i=n}^{\infty} t_i x^i,$$

where $n \in \mathbf{Z}$, each $t_i \in R$ and $t_n \neq 0$. The integer n is called the *order* of t, and the element t_n is called the *leading coefficient* of t, which we shall denote by $\lambda(t)$. By convention, $\lambda(0) = 0$. For a right ideal I of T, define $\lambda(I) = \{\lambda(t) | t \in I\}$, and observe that $\lambda(I)$ is a right ideal of R.

LEMMA 1. *Let R be a ring, let $T = R((x))$ and let I, J be right ideals of T such that $I \subseteq J$. If $\lambda(I) = \lambda(J)$ and $\lambda(I)$ is finitely generated, then $I = J$.*

PROOF. We may assume that $I \neq 0$. Choose nonzero elements $a_1, \ldots, a_m$ in I such that $\lambda(a_1), \ldots, \lambda(a_m)$ generate $\lambda(I)$. After multiplying the a_i by suitable powers of x, we may assume that the a_i all have order 0.

Now consider any nonzero element $b \in J$. In showing that $b \in I$, there is no harm in multiplying b by a power of x. Hence, we may assume that b has order 0. We construct elements $s_{ij} \in R$ (for $i = 1, \ldots, m$ and $j = 0, 1, 2, \ldots$) such that for each $n = 0, 1, 2, \ldots,$ the element

$$b - \sum_{i=1}^{m} \sum_{j=0}^{n} a_i s_{ij} x^j$$

has order greater than n.

Since $\lambda(b) \in \lambda(J) = \lambda(I) = \Sigma\lambda(a_i)R$, there exist elements $s_{i0} \in R$ such that

$$\lambda(b) = \lambda(a_1)s_{10} + \cdots + \lambda(a_m)s_{m0}.$$

As $a_1, \ldots, a_m$, b all have order 0, the element

$$b - (a_1 s_{10} + \cdots + a_m s_{m0})$$

must have order greater than 0.

Now assume that we have constructed $s_{ij} \in R$ for $i = 1, \ldots, m$ and $j = 0, 1, \ldots, n$ such that the element

$$c = b - \sum_{i=1}^{m} \sum_{j=0}^{n} a_i s_{ij} x^j$$

has order greater than n. Note that $c \in J$, and let c_{n+1} denote the coefficient of x^{n+1} in c. Either $c_{n+1} = 0$ or $c_{n+1} = \lambda(c)$, whence $c_{n+1} \in \lambda(J)$ in either case. There exist elements $s_{i,n+1} \in R$ such that

$$c_{n+1} = \lambda(a_1)s_{1,n+1} + \cdots + \lambda(a_m)s_{m,n+1},$$

and the element

$$c - \left(a_1 s_{1,n+1}x^{n+1} + \cdots + a_m s_{m,n+1}x^{n+1}\right)$$

has order greater than $n + 1$. This completes the induction step.

Finally, setting $d_i = \sum_{j=0}^{\infty} s_{ij}x^j$ for each $i = 1,\ldots,m$, we conclude that $b = a_1 d_1 + \cdots + a_m d_m$. Therefore $b \in I$. □

PROPOSITION 2. *Let R be a right noetherian ring, and let $T = R((x))$. Then T is a right noetherian ring and* r.K.dim.(T) = r.K.dim.(R).

PROOF. We have a map λ from the lattice of right ideals of T to the lattice of right ideals of R, and Lemma 1 shows that λ preserves strict inclusions. Consequently, T is right noetherian, and r.K.dim.$(T) \leqslant$ r.K.dim.(R).

For each right ideal I of R, let $I((x))$ denote the right ideal of T consisting of those elements of T with all coefficients lying in I. The map $I \mapsto I((x))$ defines an embedding of the lattice of right ideals of R into the lattice of right ideals of T, whence r.K.dim.$(R) \leqslant$ r.K.dim.(T). □

THEOREM 3. *Let R be a right noetherian ring, and let $T = R((x))$. Then* r.gl.dim.(T) = r.gl.dim.(R).

PROOF. Since R is right coherent, all direct products of flat left R-modules are flat [**3**, Theorem 2.1]. Hence, for each integer n, the set T_n consisting of those elements of T with order at least n is a flat left R-module. Thus T, being the union of the T_n, is flat as a left R-module. In addition, R is an (R, R)-bimodule direct summand of T. Therefore r.gl.dim.$(R) \leqslant$ r.gl.dim(T), by [**5**, Lemma 1].

We may now assume that r.gl.dim.$(R) = n < \infty$. Set $S = R[[x]]$. Since x is a central regular element in the Jacobson radical of S and $S/xS \cong R$, [**4**, Part III, Theorem 10] shows that r.gl.dim$(S) = n + 1$. On the other hand, r.gl.dim.$(T) \leqslant$ r.gl.dim(S) because T is a central localization of S. Thus r.gl.dim.(T) equals either n or $n + 1$.

If r.gl.dim$(T) = n + 1$, there exists a right ideal I in T such that T/I has projective dimension $n + 1$. Set $J = I \cap S$ and $A = S/J$, and observe that $A \otimes_S T \cong T/I$. Now A is a finitely generated right S-module on which x is a non-zero-divisor, and we observe that A must have projective dimension $n + 1$. According to [**4**, Part III, Theorem 9], the right R-module A/Ax must have projective dimension $n + 1$, which is impossible.

Therefore r.gl.dim.$(T) = n$. □

REFERENCES

1. S. A. Amitsur, *Prime rings having polynomial identities with arbitrary coefficients*, Proc. London Math. Soc. (3) **17** (1967), 470–486.

2. K. A. Brown and R. B. Warfield, Jr., *Krull and global dimensions of fully bounded noetherian rings*, Proc. Amer. Math. Soc. **92** (1984), 169–174.

3. S. U. Chase, *Direct products of modules*, Trans. Amer. Math. Soc. **97** (1960), 457–473.
4. I. Kaplansky, *Fields and rings*, Univ. of Chicago Press, Chicago, Ill., 1969.
5. J. C. McConnell, *On the global dimension of some rings*, Math. Z. **153** (1977), 253–254.
6. C. Procesi, *Rings with polynomial identities*, Dekker, New York, 1973.
7. R. Resco, L. W. Small and J. T. Stafford, *Krull and global dimensions of semiprime noetherian PI-rings*, Trans. Amer. Math. Soc. **274** (1982), 285–295.

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