PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 92, Number 2, October 1984

FLAT COVERS AND FLAT COTORSION MODULES

EDGAR ENOCHS

ABSTRACT. It is not known whether modules over an arbitrary ring have flat covers, however for certain modules over commutative noetherian rings they can be shown to exist. These covers, in turn, have an interesting connection with flat cotorsion modules. A complete description of flat cotorsion modules analogous to that given by Harrison for torsion free, cotorsion abelian groups will be given.

In this article, R will denote a commutative noetherian ring.

NOTATION. If R is a local ring, $m(R)$ will denote its maximal ideal. For $p \in \operatorname{Spec}(R)$, $\hat{R}_p$ will denote the completion of the local ring R_p, and for any R-module, $\hat{M}_p$ will denote the (separated) completion of the R_p module M_p with the $m(R_p)$-adic topology. $k(p)$ denotes the residue field of R_p ($\cong \hat{R}_p/m(\hat{R}_p)$). $E(M)$ will be an injective envelope of M. If R is local, M^{ν} denotes the Matlis dual, $\operatorname{Hom}(M, E(R/m(R)))$, of M, and M is said to be reflexive if $M^{\nu\nu} \cong M$ naturally. For a set X, M^X is the module of all functions $X \to M$ and $M^{(X)}$ the submodule of those functions with finite support. $\operatorname{Soc}(M)$ will denote the socle of M.

1. Flat covers.

DEFINITION (SEE [1]). A linear map $\phi\colon F \to M$ is said to be a flat precover of M if F is flat and if any diagram

$$\begin{array}{ccc} G & & \\ \downarrow & \searrow & \\ F & \xrightarrow[\phi]{} & M \end{array}$$

where G is flat can be completed to a commutative diagram (i.e. $\operatorname{Hom}(G, F) \to \operatorname{Hom}(G, M)$ is surjective). If, furthermore

$$\begin{array}{ccc} F & & \\ \downarrow & \searrow \phi & \\ F & \xrightarrow[\phi]{} & M \end{array}$$

can only be completed by automorphisms of F, we say F is a flat cover of M (with ϕ understood). If M has a flat precover then it has a cover [**1**, Theorem 3.1]. If it has a cover, it is clearly unique. If R is Prüfer they always exist and coincide with the torsion free covers shown to exist in [**2**, Theorems 1, 2]. For example, if k is a field, $k[[x]] \to k$ (evaluation at 0) is a flat and torsion free cover of the $k[x]$-module k

Received by the editors October 27, 1983.
1980 *Mathematics Subject Classification*. Primary 13C05, 13C11.

(with $xk = 0$) (see [**2** or **3**]). If $n \geqslant 2$, then $k[[x_1,\ldots,x_n]] \to k$, as noted in [**4**], is not a torsion free cover, but will be shown to be a flat cover below.

If ϕ': $F' \to M$ is a flat precover, ϕ: $F \to M$ a flat cover and $\phi \circ f = \phi'$, then f is surjective and $\operatorname{Ker}(f)$ is a direct summand of F'. Hence we have:

LEMMA 1.1. *A flat precover* ϕ: $F \to M$ *is a cover if and only if* $\operatorname{Ker}(\phi)$ *contains no nonzero direct summand of* F.

The following are easy:

LEMMA 1.2. $M = \prod M_i$ *has a flat cover if and only if each* M_j *does.*

PROOF. If $F_i \to M_i$ is a flat cover for each i then $\prod F_k \to \prod M_i$ is a flat precover, so, as noted above, $\prod M_i$ has a flat cover. If $F \to \prod M_i$ is a flat cover then for any j, $F \to \prod M_i \to M_j$ is a flat precover, so M_j has a cover.

LEMMA 1.3. *If* ϕ: $F \to M$ *is a flat cover and* $F = F_1 \oplus F_2$, $M = M_1 \oplus M_2$ *are decompositions compatible with* ϕ (*i.e.* $\phi(F_i) \subset M_i$), *then* $F_i \to M_i$ *is a flat cover for* $i = 1, 2$.

PROOF. Easily $F_i \to M_i$ is a precover. By Lemma 1.1, it is also a cover.

REMARK. If each $F_i \to M_i$ is a flat cover and the index set I is infinite, $\prod F_i \to \prod M_i$ may fail to be a cover and $\oplus F_i \to \oplus M_i$ may fail to be a precover. If I is finite, $\oplus F_i \to \oplus M_i$ is a cover.

In the following we will need

LEMMA 1.4 (ISHIKAWA [**5**, THEOREM 1.5]). *If* E *and* E' *are injective modules, then* $\operatorname{Hom}(E, E')$ *is a flat* R*-module.*

PROPOSITION 1.1. *If* M *is any module and* E *is an injective module, then* $\operatorname{Hom}(E, M)$ *and* $\operatorname{Hom}(M, E)$ *have flat covers.*

PROOF. From $M \subset E(M)$ we have a map $\operatorname{Hom}(E(M), E) \to \operatorname{Hom}(M, E)$. But $\operatorname{Hom}(E(M), E)$ is flat, so we want to show that if F is flat, $\operatorname{Hom}(F, \operatorname{Hom}(E(M), E)) \to \operatorname{Hom}(F, \operatorname{Hom}(M, E))$ is a surjection, or, equivalently, that $\operatorname{Hom}(F \otimes E(M), E) \to \operatorname{Hom}(F \otimes M, E)$ is, but this is obvious, F being flat and E injective. Since $\operatorname{Hom}(M, E)$ has a flat precover, it has a flat cover.

The case $\operatorname{Hom}(E, M)$ is similar. Here we use the fact that M has an injective precover $E' \to M$ (i.e. $\operatorname{Hom}(E, E') \to \operatorname{Hom}(E, M)$ is surjective for any injective module E) (see [**1**, Proposition 2.2]). Then if F is flat, to show $\operatorname{Hom}(F, \operatorname{Hom}(E, E'))$ $\to \operatorname{Hom}(F, \operatorname{Hom}(E, M))$ (or, equivalently, $\operatorname{Hom}(F \otimes E, E') \to \operatorname{Hom}(F \otimes E, M)$) is a surjection, we need only note that $F \otimes E$ is injective.

PROPOSITION 1.2. *If* M *is an* $\hat{R}_p$*-module and Matlis reflexive, then* M *has a flat cover as an* R*-module.*

PROOF. Since $M \cong M^{\nu\nu} = \operatorname{Hom}(M^{\nu}, E(k(p)))$, M has a flat $\hat{R}_p$-module cover $F \to M$ by Proposition 1.1. If G is a flat R-module and $G \to M$ is linear, we have a factorization $G \to G \otimes \hat{R}_p \to M$. But $G \otimes \hat{R}_p$ is $\hat{R}_p$ flat, so $G \otimes \hat{R}_p \to M$ can be lifted to F. Hence $F \to M$ is a flat precover as R-modules since F is flat as an R-module.

If R is a complete local ring, finitely generated and artinian modules are known to be reflexive. If $S \subset M$ then the diagram

$$\begin{array}{ccccccccc} 0 & \to & S & \to & M & \to & M/S & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & S^{\nu\nu} & \to & M^{\nu\nu} & \to & (M/S)^{\nu\nu} & \to & 0 \end{array}$$

with exact rows and injective vertical maps shows that M is reflexive if and only if S and M/S are. A direct sum $\oplus M_i$ is reflexive if and only if each M_i is reflexive and $M_i = 0$ except for a finite number of i. This gives an easy characterization of reflexive modules (which may be known, but I have not located it in the literature).

PROPOSITION 1.3. *If R is a complete local ring then M is Matlis reflexive if and only if M has a finitely generated submodule S such that M/S is artinian.*

PROOF. The "only if" follows from the above. If $M = 0$ there is nothing to prove. If $M \neq 0$ there is a finitely generated submodule $S_1 \subset M$ such that $\mathrm{Soc}(M/S_1) \neq 0$. If $\mathrm{Soc}(M/S_1)$ is essential in M/S_1, it is well known that M/S_1 (and in fact $E(M/S_1)$) is artinian. If it is not essential, let $N/S_1 \cap \mathrm{Soc}(M/S_1) = 0$ with $S_1 \subsetneq N$. Then there is a finitely generated S_2 with $S_1 \subset S_2 \subset N$ and $\mathrm{Soc}(N/S_2) \neq 0$. But then $\mathrm{Soc}(M/S_1) \to \mathrm{Soc}(M/S_2)$ is injective but not surjective. We repeat the procedure and see that it must stop, for otherwise if $T = \bigcup S_n$, $\mathrm{Soc}(M/T)$ is an infinite direct sum. This is not possible by the above.

EXAMPLE. Since $k(p)$ is a reflexive $\hat{R}_p$-module, it has a flat cover over R. Hence for any set X, $k(p)^X$ and, hence, $k(p)^{(X)}$ (it is a direct summand) has a flat cover over R.

Using the proof of Proposition 1.1 we see that a precover of $k(p)^{(X)} \cong \mathrm{Hom}(k(p), E(k(p))^{(X)})$ is $\mathrm{Hom}(E(k(p)), E(k(p))^{(X)})$. As noted in Griffith [7, p. 306, and also in Fuchs **8**, Proposition 44.3, in the case $R = Z$] the latter module is the completion of a free $\hat{R}_p$-module with base indexed by X, i.e. it is the submodule of $\hat{R}_p^X$ consisting of elements (r_x) with countable support and such that $\lim r_{x_i} = 0$ for any distinct $x_1, x_2, x_3, \ldots$ in X. If T designates this module then $T \to k(p)^{(X)}$ is a flat precover and is seen to induce an isomorphism $T/m(\hat{R}_p)T \to k(p)^{(X)}$. If a direct summand S of T is in $m(\hat{R}_p)T$ then $S = m(\hat{R}_p)S$. Since T (and even $\hat{R}_p^X$) is separated in the $m(\hat{R}_p)$-adic topology, this is impossible unless $S = 0$. So by Lemma 1.1 we have a cover.

Any decomposition $T = T_1 \oplus T_2$ gives one of $T/m(\hat{R}_p)T$, so by Lemma 1.3, $T_1 \to T_1/m(\hat{R}_p)T_1$ is a flat cover. Uniqueness of covers and the computation above show that T_1 is also the completion of a free $\hat{R}_p$-module whose dimension is the same as that of $T_1/m(\hat{R}_p)T_1$ over $k(p)$. This means that a direct summand of the completion of a free module is again such.

If k is a field and $\mathrm{Card}(X) = m < \infty$, the construction above shows that $k[[x_1, \ldots, x_n]]^m \to k^m$ is a flat cover. If X is infinite, $k[[x_1, \ldots, x_n]]^{(X)} \to k^{(X)}$ is not even a precover.

2. Cotorsion modules.

DEFINITION. A module M is said to be cotorsion if $\operatorname{Ext}^1(F, M) = 0$ for all flat modules F. This generalizes the definition for abelian groups. It differs from Matlis' definition in his deep study [**9**] but his concern was with domains. It agrees with Fuchs' more general definition in [**10**] which deals with torsion theories.

Cotorsion groups are known to be uniquely, up to isomorphism, the direct sum of a divisible group, a cotorsion group which has no torsion free direct summands and a torsion free cotorsion group. The latter were classified by Harrison [**13**] as products $G = \prod T_p$ over all primes p where T_p is a direct summand of $\hat{Z}_p^X$ for some X. Furthermore, G is uniquely determined by the dimensions of T_p/pT_p over $Z/(p)$. Our object is to extend this characterization to modules over any R.

We need the standard

LEMMA 2.1. *If E is an injective module and M any module then* $\operatorname{Hom}(M, E)$ *is pure injective* (*and so cotorsion*).

PROOF. Pure injective means that if $0 \to S \to N$ is a pure injection, $H(N, \operatorname{Hom}(M, E)) \to \operatorname{Hom}(S, \operatorname{Hom}(M, E)) \to 0$ is surjective, but this is a result of the standard identity.

Note that if M is $\hat{R}_p$-pure injective then M is R-pure injective. Hence we see that any direct summand of a product of modules each of which is reflexive over some $\hat{R}_p$ is cotorsion (these modules also have flat covers).

LEMMA 2.2. *If $\phi\colon F \to M$ is a flat cover then* $\operatorname{Ker}(\phi)$ *is cotorsion.*

PROOF. If $S \subset N$ is a submodule for some module N with N/S flat and f: $S \to \operatorname{Ker}(\phi)$ is linear, let $N \oplus_f F$ be the amalgamated sum of N and F along S. Then $F \subset N \oplus_f F$ and $N \oplus_f F$ is flat. ϕ can be extended to a map $N \oplus_f F \to M$ which maps N to 0. If we complete

$$\begin{array}{ccc} N \oplus_f F & & \\ \downarrow & \searrow & \\ F & \underset{\phi}{\to} & M \end{array}$$

we can assume the vertical map induced the identity on F. But then the vertical map gives a map $N \to \operatorname{Ker}(\phi)$ which extends f. This clearly implies $\operatorname{Ker}(\phi)$ is cotorsion.

COROLLARY. *If M is cotorsion then so is F.*

PROOF. $0 = \operatorname{Ext}^1(G, \operatorname{Ker}(\phi)) \to \operatorname{Ext}^1(G, F) \to \operatorname{Ext}^1(G, M)$ is exact.

LEMMA 2.3. *A flat module F is cotorsion if and only if it is a direct summand of a module* $\operatorname{Hom}(E, E')$ *where E and E' are injective.*

PROOF. A direct summand of $\operatorname{Hom}(E, E')$ is cotorsion by Lemma 2.1 and flat by Lemma 1.4. Conversely, if F is flat, let E be an injective generator. Then it is standard that $F \to \operatorname{Hom}(\operatorname{Hom}(F, E), E)$ is a pure injection (see [**11** or **12**]). $\operatorname{Hom}(F, E)$ is injective so $\operatorname{Hom}(\operatorname{Hom}(F, E), E)$ is flat. F being cotorsion and a pure submodule makes it a direct summand.

We now investigate the modules Hom(E, E'). By Matlis [6], E can be written $\oplus E(k(p))^{(X_p)}$ (over p) so

$$\mathrm{Hom}(E, E') \cong \prod \mathrm{Hom}\big(E(k(p)), E'\big)^{X_p} \cong \prod \mathrm{Hom}\big(E(k(p)), E'^{X_p}\big);$$

hence we consider the modules Hom$(E(k(p)), G)$ for G injective. We have

$$\mathrm{Hom}(E(k(p)), G) \cong \mathrm{Hom}\big(E(k(p)) \otimes R_p, G\big) \cong \mathrm{Hom}\big(E(k(p)), \mathrm{Hom}(R_p, G)\big).$$

But Hom(R_p, G) is injective over R_p so is a sum of $E(k(q))$'s for $q \subset p$. But Hom$(E(k(p)), E(k(q))) = 0$ if $q \subsetneq p$. We conclude that

$$\mathrm{Hom}(E(k(p)), G) \cong \mathrm{Hom}\big(E(k(p)), E(k(p))^{(Y)}\big)$$

for some Y, i.e. it is the completion of a free $\hat{R}_p$-module with base indexed by Y.

We now have

THEOREM. *For a module F the following are equivalent*:

(a) *F is a flat cover of some cotorsion module.*

(b) *F is flat and cotorsion.*

(c) *$F \cong \prod T_p$ (over $p \in$ Spec(R)) where T_p is the completion of a free module over R_p.*

Furthermore the decomposition in (c) *is uniquely determined by the dimension of the free modules.*

PROOF. (a) $\Rightarrow$ (b) By the corollary to Lemma 2.2.

(b) $\Rightarrow$ (c) We note that we have argued above that F is a direct summand of such a product $\prod T_p$. If $G = \prod T_p$, then if $q \in$ Spec(R) we have $qT_p = T_p$ if $q \not\subset p$ and $\bigcap q^n T_p = 0$ if $q \subset p$. Hence $G' = \bigcap q^n T_p = \prod T_p$ for $p \not\subset q$, so $H = G/G' \cong \prod T_p$ for $p \subset q$. But then

$$\bigcap_{p \subsetneq q} \Big(\bigcap_n p^n H \Big) \cong T_q.$$

This means that given G we can "recover" each T_p. The procedure commutes with direct sums, i.e. if $G = G_1 \oplus G_2$ then we get an induced decomposition $T_p = (T_p)_1 \oplus (T_p)_2$ for each p so that $G_1 \cong \prod (T_p)_1$. But as noted earler, $(T_p)_1$ is again the completion of a free module over R_p. This proves (b) $\Rightarrow$ (c) and the last statement of the theorem.

For (c) $\Rightarrow$ (a), we already have that $T_p \to T_p/m(\hat{R}_p)T_p$ is a flat cover for each p so $\prod T_p \to \prod T_p/m(\hat{R}_p)T_p$ is a flat precover with kernel $K = \prod m(\hat{R}_p)T_p$. Let $F = \prod T_p$ and suppose $S \subset K$ is a direct summand of F. If q is such that $S \subset qF$ then $S = qS$, which implies that the projection of S on T_q is 0 (since T_q is separated in the q-adic topology). Hence $S = 0$ if $S \subset qF$ for all q. If not, let q be maximal with $S \not\subset qF$. But if $q \not\subset p$, then $qT_p = T_p$. If $q \subsetneq p$ then, as above, the projection of S on T_p is 0. Since $qT_q = m(\hat{R}_q)T_q$ we get $S \subset qF$, which is a contradiction.

REFERENCES

1. E. Enochs, *Injective and flat covers, envelopes and resolvents*, Israel J. Math. **39** (1981), 189–209.

2. ______, *Torsion free covering modules*, Proc. Amer. Math. Soc. **14** (1963), 884–889.

3. B. Banaschewski, *On coverings of modules*, Math. Nachr. **31** (1966), 57–71.

4. E. Enochs, *Torsion free covering modules*. II, Arch. Math. (Basel) **22** (1971), 37–52.

5. T. Ishikawa, *On injective modules and flat modules*, J. Math. Soc. Japan **17** (1965), 291–296.

6. E. Matlis, *Injective modules over noetherian rings*, Pacific J. Math. **8** (1958), 511–528.

7. P. Griffith, *A representation theorem for complete local rings*, J. Pure Appl. Algebra **7** (1976), 303–315.

8. L. Fuchs, *Infinite abelian groups*, Vol. 1, Academic Press, New York, 1970.

9. E. Matlis, *Cotorsion modules*, Mem. Amer. Math. Soc. **49** (1964).

10. L. Fuchs, *Cotorsion modules over noetherian hereditary rings*, Houston J. Math. **3** (1977), 33–46.

11. M. Raymond and L. Gruson, *Critères de platitude et de projectivité*, Invent. Math. **13** (1971), 1–89.

12. R. B. Warfield, *Purity and algebraic compactness for modules*, Pacific J. Math. **28** (1969), 699–719.

13. D. K. Harrison, *Infinite abelian groups and homological methods*, Ann. of Math. (2) **69** (1959), 366–391.

Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506