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REMARKS ON THE PARAMETRIZED SYMBOL CALCULUS

MICHIO KINOSHITA

ABSTRACT. In his paper, L. Hörmander has used the Weyl calculus to study the Fourier integral operator theory. In the present paper, the author considers the correspondences W_τ, $\tau \in R$ (R is the set of the real numbers), which mean the standard correspondence of symbol and operator if $\tau = 0$, and the correspondence of Weyl type if $\tau = 1/2$, and shows the explicit asymptotic formula which describes the deviation of $W_\sigma(W_\tau)^{-1}$ from the automorphisms as Lie algebra, and makes some remarks on the above formula.

1. Symbol classes.

NOTATION.

$$p_{(\beta)}^{(\alpha)}(x,\xi) = \partial_\xi^\alpha D_x^\beta p(x,\xi),$$

where $p(x,\xi) \in C^\infty(R_x^n \times R_\xi^n)$, $D_{x_j} = -i\partial/\partial x_j$,

$$p_{(\beta,\beta')}^{(\alpha,\alpha')}(x,\xi,x',\xi') = \partial_\xi^\alpha \partial_{\xi'}^{\alpha'} D_x^\beta D_{x'}^{\beta'} p(x,\xi,x',\xi'),$$

where $p(x,\xi,x',\xi') \in C^\infty(R_x^n \times R_\xi^n \times R_{x'}^n \times R_{\xi'}^n)$, $\langle\xi\rangle = \sqrt{1+|\xi|^2}$, $\langle\xi;\xi'\rangle = \sqrt{1+|\xi|^2+|\xi'|^2}$ and $\hat{u}(\xi)$ is the Fourier transform of $u(x)$.

We denote by $S_{\rho,\delta}^m$, for any real numbers m, ρ, δ such that $0 \le \delta \le \rho \le 1$, $\delta < 1$, the set of smooth functions $p(x,\xi)$ on $R_x^n \times R_\xi^n$ which satisfy the condition that for any multi-indices α, β, there exists a constant $C_{\alpha,\beta}$ such that

$$|p_{(\beta)}^{(\alpha)}(x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{m+\delta|\beta|-\rho|\alpha|}.$$

Let $S_{\rho,\delta}^\infty$ be the set $\bigcup_{m\in\mathbf{R}} S_{\rho,\delta}^m$.

We denote by $S_{\rho,\delta}^{m,m'}$, for any real numbers m, m', ρ, δ such that $0 \le \delta \le \rho \le 1$, $\delta < 1$, the set of smooth functions $p(x,\xi,x',\xi')$ on $R_x^n \times R_\xi^n \times R_{x'}^n \times R_{\xi'}^n$, which satisfy the condition that for any multi-indices $\alpha, \beta, \alpha', \beta'$, there exists a constant $C_{\alpha,\beta,\alpha',\beta'}$ such that

$$|p_{(\beta,\beta')}^{(\alpha,\alpha')}(x,\xi,x',\xi')| \le C_{\alpha,\beta,\alpha',\beta'}\langle\xi\rangle^{m+\delta|\beta|-\rho|\alpha|}\langle\xi;\xi'\rangle^{\delta|\beta'|}\langle\xi\rangle^{m'-\rho|\alpha'|}.$$

We denote by $\mathcal{S}$ the set of the rapidly decreasing functions, and we denote by $\mathrm{Op}(S_{\rho,\delta}^m)$ the set of pseudo-differential operators which is defined by

$$(B(p)u)(x) = \int e^{ix\cdot\xi}p(x,\xi)\hat{u}(\xi)\,d\xi, \qquad p \in S_{\rho,\delta}^m,\ u \in \mathcal{S}.$$

In like manner, we denote by $\mathrm{Op}(S_{\rho,\delta}^{m,m'})$ the set of pseudo-differential operators

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which is defined by

$$(W(p)u)(x) = \iiint e^{i(x-x')\cdot\xi + ix'\cdot\xi'} p(x,\xi,x',\xi')\hat{u}(\xi')\,d\xi\,dx'\,d\xi',$$

$$p \in S^{m,m'}_{\rho,\delta}, \quad u \in \mathcal{S}.$$

We denote by R^m_τ, for any real numbers m and τ, the linear mapping from $S^m_{\rho,\delta}$ to $S^{m,0}_{\rho,\delta}$, defined by

$$(R^m_\tau(p))(x,\xi,x',\xi') = p((1-\tau)x + \tau x', \xi),$$

and we denote by W^m_τ, for any real numbers m and τ, the linear mapping from $S^m_{\rho,\delta}$ to $\mathrm{Op}(S^m_{\rho,\delta})$, defined by WR^m_τ.

REMARK 1. $\mathrm{Op}(S^{m,0}_{\rho,\delta}) = \mathrm{Op}(S^m_{\rho,\delta})$.

REMARK 2. The standard correspondence between symbols and operators is W^m_0, and the correspondence of Weyl type is $W^m_{1/2}$ (for the Weyl calculus, see [**1**]).

We denote by W^∞_τ, for any real numbers τ, the linear mapping from $S^m_{\rho,\delta}$ to $\mathrm{Op}(S^\infty_{\rho,\delta})$, defined by $W^\infty_\tau(p) = W^m_\tau(p)$, for any $p \in S^m_{\rho,\delta}$.

Let $A = W^m_\tau(p)$ and $B = W^{m'}_\tau(q)$, where $p \in S^m_{\rho,\delta}$, $q \in S^{m'}_{\rho,\delta}$. The product AB is contained in $\mathrm{Op}(S^{m+m'}_{\rho,\delta})$. Therefore, there exists a function $r \in S^{m+m'}_{\rho,\delta}$ such that $AB = W^{m+m'}_\tau(r)$. This symbol r is expressed by the formula

$$r(x,\xi) = [(\mathrm{Exp}\, i((1-\tau)D_\xi D_y - \tau D_x D_\eta))p(x,\xi)q(y,\eta)]_{(x,\xi)=(y,\eta)}.$$

We use the notation $r = p \circ_\tau q$ in the following section. This notation is used in [**3**].

2. Main result. In this section we examine the linear transformation of $\mathrm{Op}(S^m_{\rho,\delta})$ given by $K^m_{\sigma,\tau} = W^m_\sigma(W^m_\tau)^{-1}$, when $\sigma \neq \tau$.

REMARK 3. The fact that $K^\infty_{\sigma,\tau}$ is not an automorphism of $\mathrm{Op}(S^\infty_{\rho,\delta})$ as algebra is reduced to the fact that $a \circ_\sigma b \neq a \circ_\tau b$ by the composition formula above.

In this section, we consider the deviation of $K^m_{\sigma,\tau}$ from the automorphisms of $\mathrm{Op}(S^m_{\rho,\delta})$ as a Lie algebra.

REMARK 4. When $m \leq \rho - \delta$, $\mathrm{Op}(S^m_{\rho,\delta})$ is a Lie algebra. By a trivial computation, we obtain the following fact. When $A \in \mathrm{Op}(S^m_{\rho,\delta})$ and $B \in \mathrm{Op}(S^m_{\rho,\delta})$, we get

$$[K^m_{\sigma,\tau}(A), K^m_{\sigma,\tau}(B)] - K^m_{\sigma,\tau}([A,B]) = W^{2m}_\sigma(a \circ_\sigma b - b \circ_\sigma a - a \circ_\tau b + b \circ_\tau a),$$

where $a = (W^m_\tau)^{-1}A$, $b = (W^m_\tau)^{-1}B$.

We denote by $H_n(\sigma)$ the function $\sum_{k=1}^n F^{n-k}G^{k-1}$, where $F = (1-\sigma)\xi\cdot y - \sigma x\eta$, $G = (1-\sigma)x\cdot\eta - \sigma\xi\cdot y$.

REMARK 5. This function has an invariant property with respect to the changing of $\xi\cdot y$ and $x\cdot\eta$. Obvious calculation gives

$$H_n(\sigma) = \sum_{k=1}^{n}\sum_{r=0}^{n-k}\sum_{s=0}^{k-1}(-1)^{n-k-r+s}\binom{n-k}{r}\binom{k-1}{s} \cdot (1-\sigma)^{k+r-s-1}\sigma^{n+s-k-r}(\xi\cdot y)^{r+s}(x\cdot\eta)^{n-(r+s)-1}.$$

We denote by $T_k(D)$ the operator corresponding to

$$T_k(x,\xi,y,\eta) = \frac{i^k}{k!}(\xi y - x\eta)(H_k(\sigma) - H_k(\tau)).$$

The deviation is described by the following theorem.

THEOREM. *If $a \in S^{m_1}_{\rho,\delta}$ and $b \in S^{m_2}_{\rho,\delta}$, then*

$$a \circ_\sigma b - b \circ_\sigma a - a \circ_\tau b + b \circ_\tau a - \sum_{k=0}^{n} T_k(D)a(x,\xi)b(y,\eta)|_{(x,\xi)=(y,\eta)} \in S^{m_1+m_2-(n+1)(\rho-\delta)}_{\rho,\delta}.$$

In the case of $k = 0, 1, 2, 3$, we have

$$T_0(x,\xi,y,\eta) = 0, \quad T_1(x,\xi,y,\eta) = 0,$$
$$T_2(x,\xi,y,\eta) = (\sigma-\tau)(\xi y - x\eta)(\xi y + x\eta),$$
$$T_3(x,\xi,y,\eta) = -\frac{i}{2}(\sigma-\tau)(\xi y - x\eta)(\sigma+\tau-1)(\xi y + x\eta)^2.$$

PROOF. Essentially the product formula and calculations give the proof.

REMARK 6. $T_k(x,\xi,y,\eta)$ is divisible by $(\sigma-\tau)(\xi y - x\eta)$. Consequently

$$T_k(x,\xi,y,\eta) = (\sigma-\tau)(\xi y - x\eta)U_k(x,\xi,y,\eta,\sigma,\tau),$$

where U_k is a symmetric function of $\xi \cdot y$ and $x \cdot \eta$, and also a symmetric function of σ and τ. By the fact that $T_2 \neq 0$, we obtain that $K^m_{\sigma,\tau}$, $m \leq \rho - \delta$, is not an automorphism of the Lie algebra $\mathrm{Op}(S^m_{\rho,\delta})$. By the fact that $T_0 = T_1 = 0$, we obtain that $\tilde{K}^m_{\sigma,\tau}$, $m \leq \rho-\delta$, which is induced by $K^m_{\sigma,\tau}$, $m \leq \rho - \delta$, is an automorphism of the Lie algebra $\mathrm{Op}(S^m_{\rho,\delta})/\mathrm{Op}(S^{2m-2(\rho-\delta)}_{\rho,\delta})$.

REMARK 7. From the property that $H_k(1-\sigma) = (-1)^{k-1}H_k(\sigma)$, we obtain that if $\sigma + \tau = 1$ and k is odd, then $T_k(x,\xi,y,\eta) = 0$.

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DEPARTMENT OF MATHEMATICS, SHIMANE UNIVERSITY, NISHI-KAWATSU 1060, MATSUE, JAPAN 690