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REMARKS ON THE PARAMETRIZED SYMBOL CALCULUS

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ABSTRACT. In his paper, L. Hörmander has used the Weyl calculus to study the Fourier integral operator theory. In the present paper, the author considers the correspondences W_τ, $\tau \in R$ (R is the set of the real numbers), which mean the standard correspondence of symbol and operator if $\tau = 0$, and the correspondence of Weyl type if $\tau = 1/2$, and shows the explicit asymptotic formula which describes the deviation of $W_\sigma(W_\tau)^{-1}$ from the automorphisms as Lie algebra, and makes some remarks on the above formula.

1. Symbol classes.

NOTATION.

$$p_{(\beta)}^{(\alpha)}(x,\xi) = \partial_\xi^\alpha D_x^\beta p(x,\xi),$$

where $p(x,\xi) \in C^\infty(R_x^n \times R_\xi^n)$, $D_{x_j} = -i\partial/\partial x_j$,

$$p_{(\beta,\beta')}^{(\alpha,\alpha')}(x,\xi,x',\xi') = \partial_\xi^\alpha \partial_{\xi'}^{\alpha'} D_x^\beta D_{x'}^{\beta'} p(x,\xi,x',\xi'),$$

where $p(x,\xi,x',\xi') \in C^\infty(R_x^n \times R_\xi^n \times R_{x'}^n \times R_{\xi'}^n)$, $\langle\xi\rangle = \sqrt{1+|\xi|^2}$, $\langle\xi;\xi'\rangle = \sqrt{1+|\xi|^2+|\xi'|^2}$ and $\hat{u}(\xi)$ is the Fourier transform of $u(x)$.

We denote by $S_{\rho,\delta}^m$, for any real numbers m,ρ,δ such that $0 \le \delta \le \rho \le 1$, $\delta < 1$, the set of smooth functions $p(x,\xi)$ on $R_x^n \times R_\xi^n$ which satisfy the condition that for any multi-indices α,β, there exists a constant $C_{\alpha,\beta}$ such that

$$|p_{(\beta)}^{(\alpha)}(x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{m+\delta|\beta|-\rho|\alpha|}.$$

Let $S_{\rho,\delta}^\infty$ be the set $\bigcup_{m\in\mathbf{R}} S_{\rho,\delta}^m$.

We denote by $S_{\rho,\delta}^{m,m'}$, for any real numbers m,m',ρ,δ such that $0 \le \delta \le \rho \le 1$, $\delta < 1$, the set of smooth functions $p(x,\xi,x',\xi')$ on $R_x^n \times R_\xi^n \times R_{x'}^n \times R_{\xi'}^n$, which satisfy the condition that for any multi-indices $\alpha,\beta,\alpha',\beta'$, there exists a constant $C_{\alpha,\beta,\alpha',\beta'}$ such that

$$|p_{(\beta,\beta')}^{(\alpha,\alpha')}(x,\xi,x',\xi')| \le C_{\alpha,\beta,\alpha',\beta'}\langle\xi\rangle^{m+\delta|\beta|-\rho|\alpha|}\langle\xi;\xi'\rangle^{\delta|\beta'|}\langle\xi\rangle^{m'-\rho|\alpha'|}.$$

We denote by $\mathcal{S}$ the set of the rapidly decreasing functions, and we denote by $\mathrm{Op}(S_{\rho,\delta}^m)$ the set of pseudo-differential operators which is defined by

$$(B(p)u)(x) = \int e^{ix\cdot\xi}p(x,\xi)\hat{u}(\xi)\,d\xi, \qquad p \in S_{\rho,\delta}^m,\ u \in \mathcal{S}.$$

In like manner, we denote by $\mathrm{Op}(S_{\rho,\delta}^{m,m'})$ the set of pseudo-differential operators

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which is defined by

$$(W(p)u)(x) = \iiint e^{i(x-x')\cdot\xi + ix'\cdot\xi'} p(x,\xi,x',\xi')\hat{u}(\xi')\,d\xi\,dx'\,d\xi',$$

$$p \in S^{m,m'}_{\rho,\delta},\ u \in \mathcal{S}.$$

We denote by R^m_τ, for any real numbers m and τ, the linear mapping from $S^m_{\rho,\delta}$ to $S^{m,0}_{\rho,\delta}$, defined by

$$(R^m_\tau(p))(x,\xi,x',\xi') = p((1-\tau)x + \tau x', \xi),$$

and we denote by W^m_τ, for any real numbers m and τ, the linear mapping from $S^m_{\rho,\delta}$ to $\mathrm{Op}(S^m_{\rho,\delta})$, defined by WR^m_τ.

REMARK 1. $\mathrm{Op}(S^{m,0}_{\rho,\delta}) = \mathrm{Op}(S^m_{\rho,\delta})$.

REMARK 2. The standard correspondence between symbols and operators is W^m_0, and the correspondence of Weyl type is $W^m_{1/2}$ (for the Weyl calculus, see [**1**]).

We denote by W^∞_τ, for any real numbers τ, the linear mapping from $S^m_{\rho,\delta}$ to $\mathrm{Op}(S^\infty_{\rho,\delta})$, defined by $W^\infty_\tau(p) = W^m_\tau(p)$, for any $p \in S^m_{\rho,\delta}$.

Let $A = W^m_\tau(p)$ and $B = W^{m'}_\tau(q)$, where $p \in S^m_{\rho,\delta}$, $q \in S^{m'}_{\rho,\delta}$. The product AB is contained in $\mathrm{Op}(S^{m+m'}_{\rho,\delta})$. Therefore, there exists a function $r \in S^{m+m'}_{\rho,\delta}$ such that $AB = W^{m+m'}_\tau(r)$. This symbol r is expressed by the formula

$$r(x,\xi) = [(\mathrm{Exp}\, i((1-\tau)D_\xi D_y - \tau D_x D_\eta))p(x,\xi)q(y,\eta)]_{(x,\xi)=(y,\eta)}.$$

We use the notation $r = p \circ_\tau q$ in the following section. This notation is used in [**3**].

2. Main result. In this section we examine the linear transformation of $\mathrm{Op}(S^m_{\rho,\delta})$ given by $K^m_{\sigma,\tau} = W^m_\sigma (W^m_\tau)^{-1}$, when $\sigma \neq \tau$.

REMARK 3. The fact that $K^\infty_{\sigma,\tau}$ is not an automorphism of $\mathrm{Op}(S^\infty_{\rho,\delta})$ as algebra is reduced to the fact that $a \circ_\sigma b \neq a \circ_\tau b$ by the composition formula above.

In this section, we consider the deviation of $K^m_{\sigma,\tau}$ from the automorphisms of $\mathrm{Op}(S^m_{\rho,\delta})$ as a Lie algebra.

REMARK 4. When $m \leq \rho - \delta$, $\mathrm{Op}(S^m_{\rho,\delta})$ is a Lie algebra. By a trivial computation, we obtain the following fact. When $A \in \mathrm{Op}(S^m_{\rho,\delta})$ and $B \in \mathrm{Op}(S^m_{\rho,\delta})$, we get

$$[K^m_{\sigma,\tau}(A), K^m_{\sigma,\tau}(B)] - K^m_{\sigma,\tau}([A,B]) = W^{2m}_\sigma(a \circ_\sigma b - b \circ_\sigma a - a \circ_\tau b + b \circ_\tau a),$$

where $a = (W^m_\tau)^{-1}A$, $b = (W^m_\tau)^{-1}B$.

We denote by $H_n(\sigma)$ the function $\sum_{k=1}^n F^{n-k}G^{k-1}$, where $F = (1-\sigma)\xi\cdot y - \sigma x\eta$, $G = (1-\sigma)x\cdot\eta - \sigma\xi\cdot y$.

REMARK 5. This function has an invariant property with respect to the changing of $\xi\cdot y$ and $x\cdot\eta$. Obvious calculation gives

$$H_n(\sigma) = \sum_{k=1}^{n}\sum_{r=0}^{n-k}\sum_{s=0}^{k-1}(-1)^{n-k-r+s}\binom{n-k}{r}\binom{k-1}{s} \cdot (1-\sigma)^{k+r-s-1}\sigma^{n+s-k-r}(\xi\cdot y)^{r+s}(x\cdot\eta)^{n-(r+s)-1}.$$

We denote by $T_k(D)$ the operator corresponding to

$$T_k(x,\xi,y,\eta) = \frac{i^k}{k!}(\xi y - x\eta)(H_k(\sigma) - H_k(\tau)).$$

The deviation is described by the following theorem.

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THEOREM. *If* $a \in S^{m_1}_{\rho,\delta}$ *and* $b \in S^{m_2}_{\rho,\delta}$, *then*

$$a \circ_\sigma b - b \circ_\sigma a - a \circ_\tau b + b \circ_\tau a - \sum_{k=0}^{n} T_k(D) a(x,\xi) b(y,\eta)|_{(x,\xi)=(y,\eta)} \in S^{m_1+m_2-(n+1)(\rho-\delta)}_{\rho,\delta}.$$

In the case of $k = 0, 1, 2, 3$, *we have*

$$T_0(x,\xi,y,\eta) = 0, \quad T_1(x,\xi,y,\eta) = 0,$$

$$T_2(x,\xi,y,\eta) = (\sigma-\tau)(\xi y - x\eta)(\xi y + x\eta),$$

$$T_3(x,\xi,y,\eta) = -\frac{i}{2}(\sigma-\tau)(\xi y - x\eta)(\sigma+\tau-1)(\xi y + x\eta)^2.$$

PROOF. Essentially the product formula and calculations give the proof.

REMARK 6. $T_k(x,\xi,y,\eta)$ is divisible by $(\sigma-\tau)(\xi y - x\eta)$. Consequently

$$T_k(x,\xi,y,\eta) = (\sigma-\tau)(\xi y - x\eta) U_k(x,\xi,y,\eta,\sigma,\tau),$$

where U_k is a symmetric function of $\xi \cdot y$ and $x \cdot \eta$, and also a symmetric function of σ and τ. By the fact that $T_2 \neq 0$, we obtain that $K^m_{\sigma,\tau}$, $m \leq \rho - \delta$, is not an automorphism of the Lie algebra $\mathrm{Op}(S^m_{\rho,\delta})$. By the fact that $T_0 = T_1 = 0$, we obtain that $\tilde{K}^m_{\sigma,\tau}$, $m \leq \rho - \delta$, which is induced by $K^m_{\sigma,\tau}$, $m \leq \rho-\delta$, is an automorphism of the Lie algebra $\mathrm{Op}(S^m_{\rho,\delta})/\mathrm{Op}(S^{2m-2(\rho-\delta)}_{\rho,\delta})$.

REMARK 7. From the property that $H_k(1-\sigma) = (-1)^{k-1} H_k(\sigma)$, we obtain that if $\sigma + \tau = 1$ and k is odd, then $T_k(x,\xi,y,\eta) = 0$.

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