REMARKS ON THE PARAMETRIZED SYMBOL CALCULUS

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ABSTRACT. In his paper, L. Hörmander has used the Weyl calculus to study the Fourier integral operator theory. In the present paper, the author considers the correspondences $\mathcal{W}_\tau$, $\tau \in \mathbb{R}$ ($\mathbb{R}$ is the set of the real numbers), which mean the standard correspondence of symbol and operator if $\tau = 0$, and the correspondence of Weyl type if $\tau = 1/2$, and shows the explicit asymptotic formula which describes the deviation of $\mathcal{W}_\sigma(\mathcal{W}_\tau)^{-1}$ from the automorphisms as Lie algebra, and makes some remarks on the above formula.

1. Symbol classes.

NOTATION.

\[ p_{(\alpha,\beta)}(x, \xi) = \partial_\xi^\alpha D_\xi^\beta p(x, \xi), \]

where $p(x, \xi) \in C^\infty(\mathbb{R}_x^m \times \mathbb{R}_\xi^n)$, $D_{x_j} = -i\partial/\partial x_j$,

\[ p_{(\alpha,\beta,\alpha')}(x, \xi, x', \xi') = \partial_\xi^\alpha \partial_{\xi'}^\alpha D_\xi^\beta D_x^{\alpha'} p(x, \xi, x', \xi'), \]

where $p(x, \xi, x', \xi') \in C^\infty(\mathbb{R}_x^m \times \mathbb{R}_\xi^n \times \mathbb{R}_x^m \times \mathbb{R}_\xi^n)$, $(\xi) = \sqrt{1 + |\xi|^2}$, $(\xi; \xi') = \sqrt{1 + |\xi|^2 + |\xi'|^2}$ and $\hat{u}(\xi)$ is the Fourier transform of $u(x)$.

We denote by $\mathcal{S}^m_{m', \rho, \delta}$, for any real numbers $m, \rho, \delta$ such that $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$, the set of smooth functions $p(x, \xi)$ on $\mathbb{R}_x^m \times \mathbb{R}_\xi^n$ which satisfy the condition that for any multi-indices $\alpha, \beta$, there exists a constant $C_{\alpha, \beta}$ such that

\[ |p_{(\alpha,\beta)}(x, \xi)| \leq C_{\alpha, \beta} |\xi|^{m+\delta} |\beta| - \rho |\alpha|. \]

Let $\mathcal{S}^\infty_{\rho, \delta}$ be the set $\bigcup_{m \in \mathbb{R}} \mathcal{S}^m_{\rho, \delta}$.

We denote by $\mathcal{S}^{m,m'}_{\rho, \delta}$, for any real numbers $m, m', \rho, \delta$ such that $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$, the set of smooth functions $p(x, \xi, x', \xi')$ on $\mathbb{R}_x^m \times \mathbb{R}_\xi^n \times \mathbb{R}_x^{m'} \times \mathbb{R}_\xi^{n'}$, which satisfy the condition that for any multi-indices $\alpha, \beta, \alpha', \beta'$, there exists a constant $C_{\alpha, \beta, \alpha', \beta'}$ such that

\[ |p_{(\alpha,\beta,\alpha',\beta')}^{(\alpha,\alpha')}(x, \xi, x', \xi')| \leq C_{\alpha, \beta, \alpha', \beta'} |\xi|^{m+\delta} |\beta| - \rho |\alpha| |\xi; \xi'| |\beta'| - \rho |\alpha'|. \]

We denote by $\mathcal{S}$ the set of the rapidly decreasing functions, and we denote by $\operatorname{Op}(\mathcal{S}^m_{\rho, \delta})$ the set of pseudo-differential operators which is defined by

\[ (B(p)u)(x) = \int e^{ix \cdot \xi} p(x, \xi) \hat{u}(\xi) d\xi, \quad p \in \mathcal{S}^m_{\rho, \delta}, \ u \in \mathcal{S}. \]

In like manner, we denote by $\operatorname{Op}(\mathcal{S}^{m,m'}_{\rho, \delta})$ the set of pseudo-differential operators...
which is defined by
\[(W(p)u)(x) = \int \int \int e^{i(x-x') \xi + i x' \xi'} p(x, \xi, x', \xi') \hat{u}(\xi') d\xi dx' d\xi',\]
\[p \in S_{\rho, \delta}^{m, 0}, \ u \in S.\]

We denote by \(R_m^\tau\), for any real numbers \(m\) and \(\tau\), the linear mapping from \(S_{\rho, \delta}^m\) to \(S_{\rho, \delta}^{m, 0}\), defined by
\[(R_m^\tau(p))(x, \xi, x', \xi') = p((1 - \tau)x + \tau x', \xi),\]
and we denote by \(W_m^\tau\), for any real numbers \(m\) and \(\tau\), the linear mapping from \(S_{\rho, \delta}^m\) to \(\text{Op}(S_{\rho, \delta}^m)\), defined by \(WR_m^\tau\).

**Remark 1.** \(\text{Op}(S_{\rho, \delta}^{m, 0}) = \text{Op}(S_{\rho, \delta}^m)\).

**Remark 2.** The standard correspondence between symbols and operators is \(W_0^m\), and the correspondence of Weyl type is \(W_{1/2}^m\) (for the Weyl calculus, see [1]).

We denote by \(W_{\infty}^\tau\), for any real numbers \(\tau\), the linear mapping from \(S_{\rho, \delta}^m\) to \(\text{Op}(S_{\rho, \delta}^\infty)\), defined by \(\lim_{\tau \to \infty} W_{m}^\tau(p) = W_{\infty}^\tau(p)\), for any \(p \in S_{\rho, \delta}^m\).

Let \(A = W_m^\tau(p)\) and \(B = W_{m'}^\tau(q)\), where \(p \in S_{\rho, \delta}^m\), \(q \in S_{\rho, \delta}^{m'}\). The product \(AB\) is contained in \(\text{Op}(S_{\rho, \delta}^{m + m'})\). Therefore, there exists a function \(r \in S_{\rho, \delta}^{m + m'}\) such that \(AB = W_{m + m'}^\tau(r)\). This symbol \(r\) is expressed by the formula
\[r(x, \xi) = \left[(\text{Exp} i((1 - \tau)D \xi D_y - \tau D_x D_\eta))p(x, \xi)q(y, \eta)\right](x, \xi) = (y, \eta).

We use the notation \(r = p \circ_T q\) in the following section. This notation is used in [3].

**2. Main result.** In this section we examine the linear transformation of \(\text{Op}(S_{\rho, \delta}^m)\) given by \(K_{\sigma, \tau}^m = W_{\sigma}^m(W_{\tau}^m)^{-1}\), when \(\sigma \neq \tau\).

**Remark 3.** The fact that \(K_{\sigma, \tau}^{\infty}\) is not an automorphism of \(\text{Op}(S_{\rho, \delta}^\infty)\) as algebra is reduced to the fact that \(a \circ_\sigma b \neq a \circ_\tau b\) by the composition formula above.

In this section, we consider the deviation of \(K_{\sigma, \tau}^m\) from the automorphisms of \(\text{Op}(S_{\rho, \delta}^m)\) as a Lie algebra.

**Remark 4.** When \(m \leq \rho - \delta\), \(\text{Op}(S_{\rho, \delta}^m)\) is a Lie algebra. By a trivial computation, we obtain the following fact. When \(A \in \text{Op}(S_{\rho, \delta}^m)\) and \(B \in \text{Op}(S_{\rho, \delta}^m)\), we get
\[[K_{\sigma, \tau}^m(A), K_{\sigma, \tau}^m(B)] = K_{\sigma, \tau}^m([A, B]) = W_{\sigma}^{m + m}((a \circ_\sigma b - b \circ_\sigma a - a \circ_\sigma b + b \circ_\sigma a),\]
where \(a = (W_{\tau}^m)^{-1}A, b = (W_{\tau}^m)^{-1}B\).

We denote by \(H_n(\sigma)\) the function \(\sum_{k=1}^{n} F^{n-k}G^{k-1}\), where \(F = (1 - \sigma)\xi \cdot y - \sigma \xi \cdot y,\)
\(G = (1 - \sigma)\xi \cdot y - \sigma \xi \cdot y\).

**Remark 5.** This function has an invariant property with respect to the changing of \(\xi \cdot y\) and \(x \cdot \eta\). Obvious calculation gives
\[H_n(\sigma) = \sum_{k=1}^{n} \sum_{r=0}^{n-k} (-1)^{n-k-r+s} \binom{n-k}{r} \binom{k-1}{s} \cdot (1 - \sigma)^{k+r-s-1} \sigma^{n+s-k-r}(\xi \cdot y)^{r+s}(x \cdot \eta)^{n-(r+s)-1}.

We denote by \(T_k(D)\) the operator corresponding to
\[T_k(x, \xi, y, \eta) = \frac{i^k}{k!}(\xi y - x \eta)(H_k(\sigma) - H_k(\tau)).\]

The deviation is described by the following theorem.
THEOREM. If \(a \in S_{\rho,\delta}^{m_1}\) and \(b \in S_{\rho,\delta}^{m_2}\), then
\[
a \circ b - b \circ a - a \circ b + b \circ a - \sum_{k=0}^{n} T_k(D)a(x, \xi)b(y, \eta)(x, \xi) = (y, \eta)
\]
\[\in S_{\rho,\delta}^{m_1 + m_2 - (n+1)(\rho - \delta)}.
\]
In the case of \(k = 0, 1, 2, 3\), we have
\[
T_0(x, \xi, y, \eta) = 0, \quad T_1(x, \xi, y, \eta) = 0,
T_2(x, \xi, y, \eta) = (\sigma - \tau)(\xi y - x\eta)(\xi y + x\eta),
T_3(x, \xi, y, \eta) = -\frac{i}{2}(\sigma - \tau)(\xi y - x\eta)(\sigma + \tau - 1)(\xi y + x\eta)^2.
\]
PROOF. Essentially the product formula and calculations give the proof.
REMARK 6. \(T_k(x, \xi, y, \eta)\) is divisible by \((\sigma - \tau)(\xi y - x\eta)\). Consequently
\[
T_k(x, \xi, y, \eta) = (\sigma - \tau)(\xi y - x\eta)U_k(x, \xi, y, \eta, \sigma, \tau),
\]
where \(U_k\) is a symmetric function of \(\xi \cdot y\) and \(x \cdot \eta\), and also a symmetric function of \(\sigma\) and \(\tau\). By the fact that \(T_2 \neq 0\), we obtain that \(K_{\sigma,\tau}^m, m \leq \rho - \delta\), is not an automorphism of the Lie algebra \(\text{Op}(S_{\rho,\delta}^m)\). By the fact that \(T_0 = T_1 = 0\), we obtain that \(K_{\sigma,\tau}^m, m \leq \rho - \delta\), which is induced by \(K_{\sigma,\tau}^m, m \leq \rho - \delta\), is an automorphism of the Lie algebra \(\text{Op}(S_{\rho,\delta}^m)/\text{Op}(S_{\rho,\delta}^{2m-2(\rho - \delta)})\).
REMARK 7. From the property that \(H_k(1 - \sigma) = (-1)^{k-1}H_k(\sigma)\), we obtain that if \(\sigma + \tau = 1\) and \(k\) is odd, then \(T_k(x, \xi, y, \eta) = 0\).

REFERENCES

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