PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 92, Number 2, October 1984

LINEAR MAPS BETWEEN CERTAIN NONSEPARABLE C^*-ALGEBRAS

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ABSTRACT. There exists a noninjective commutative C^*-algebra A such that every bounded linear map of any C^*-algebra into A is decomposed as a linear combination of positive linear maps.

1. Introduction. A bounded linear map ϕ: $A \to B$ between C^*-algebras is said to be completely positive if every multiplicity map $\phi \otimes \mathrm{id}_n$: $A \otimes M_n \to B \otimes M_n$ is positive, and completely bounded if $\sup_n \|\phi \otimes \mathrm{id}_n\| < \infty$. Recently, Wittstock [**14**, Satz 4.5] proved that the linear span of completely positive maps of a unital C^*-algebra into an injective C^*-algebra is identical with the set of all completely bounded maps (see also [**6**]). We showed in [**3**] (resp. [**2**]) that given a separable C^*-algebra B, if every bounded linear (resp. completely bounded) map of any C^*-algebra into B is a linear combination of positive linear (resp. completely positive) maps, then B is finite-dimensional, namely injective.

The purpose of this paper is to study the positive decomposability for bounded linear maps in the absence of the separability assumption. The main result is that there exists a nonstonean compact Hausdorff space T such that every bounded linear map of any C^*-algebra into $C(T)$, the C^*-algebra of all continuous complex functions on T, is decomposed as a linear combination of positive linear maps. This result is used to answer negatively Smith's conjecture [**8**]. In addition, we show that every bounded linear map of any partially ordered Banach space with normal positive cone into $C_r(T)$, the Banach space of all continuous real functions on the space T, is decomposed as the difference of two positive bounded linear maps.

2. The main results. A compact Hausdorff space is said to be stonean (or extremally disconnected) if the closure of every open subset is again open. A unital C^*-algebra A is injective if and only if for any C^*-algebra B such that $B \supseteq A$, there exists a projection of B onto A of norm one [**1**, Theorem 5.3]. The set of selfadjoint elements of an injective C^*-algebra is conditionally complete [**10**, Theorem 7.1]. It is known that a compact Hausdorff space S is stonean if and only if $C(S)$ is an injective C^*-algebra (see [**4**, Chapter 3, §11, Theorems 6, 7 and **9**, Chapter III, Proposition 1.7]).

Let S_1, S_2 be stonean spaces. Suppose that each S_i contains a limit point s_i. Put $T_i = S_i - \{s_i\}$. Let T denote the space obtained from S_1 and S_2 by identifying s_1 with s_2. More precisely, T is the one-point compactification of the topological sum of locally compact spaces T_1, T_2, with the point ω at infinity. Since S_i is

Received by the editors May 23, 1983 and, in revised form, October 5, 1983.
1980 *Mathematics Subject Classification*. Primary 46L05; Secondary 46A40.
Key words and phrases. Completely bounded map, completely positive map, injective C^*-algebra, partially ordered Banach space, stonean space.

homeomorphic to $T_i \cup \{\omega\}$, we identify S_i with $T_i \cup \{\omega\}$. Then the closure of T_i is $S_i = T_i \cup \{\omega\}$. Since T_i is an open subset of T and the closure of T_i is not open, T is not stonean.

THEOREM 1. *With the above notation, every bounded selfadjoint linear map ϕ of a C^*-algebra A into $C(T)$ is decomposed as a linear combination of positive linear maps.*

PROOF. (1) We assume first that $\phi(a)(\omega) = 0$ for all a in A. For $i = 1, 2$, let ϕ_i: $A \to C(S_i)$ be defined by $\phi_i(a) = \phi(a)|S_i$, the restriction to S_i of $\phi(a)$. Since S_i is stonean, there exist positive linear maps ϕ_i^+, ϕ_i^-: $A \to C(S_i)$ such that $\phi_i = \phi_i^+ - \phi_i^-$ [**12**, Corollary 1.2.10]. We choose h_i in $C(T - T_i)$ such that $0 \le h_i \le 1$ and $h_i(\omega) = 1$. Then there exist positive linear maps ψ_i^+, ψ_i^-: $A \to C(T)$ such that for a in A,

$$\psi_i^+(a)(t) = \phi_i^+(a)(t), \quad \psi_i^-(a)(t) = \phi_i^-(a)(t) \quad \text{if } t \in S_i;$$

$$\psi_i^+(a)(t) = \phi_i^+(a)(\omega)h_i(t), \quad \psi_i^-(a)(t) = \phi_i^-(a)(\omega)h_i(t) \quad \text{if } t \in T - S_i.$$

Let $a \in A$. If $t \in S_i$, then

$$\psi_i^+(a)(t) - \psi_i^-(a)(t) = \phi_i^+(a)(t) - \phi_i^-(a)(t) = \phi_i(a)(t);$$

if $t \in T - S_i$, then

$$\psi_i^+(a)(t) - \psi_i^-(a)(t) = \phi_i^+(a)(\omega)h_i(t) - \phi_i^-(a)(\omega)h_i(t) = \phi(a)(\omega)h_i(t) = 0.$$

We put $\phi^+ = \psi_1^+ + \psi_2^+$, $\phi^- = \psi_1^- + \psi_2^-$. Both ϕ^+, ϕ^- are positive linear maps of A into $C(T)$. If $t \in S_i$, then

$$\phi^+(a)(t) - \phi^-(a)(t) = \phi_i(a)(t) = \phi(a)(t)$$

for a in A. Since $S_1 \cup S_2 = T$, we have $\phi = \phi^+ - \phi^-$.

(2) Using h in $C(T)$ such that $0 \le h \le 1$ and $h(\omega) = 1$, we define ψ: $A \to C(T)$ by $\psi(a) = \phi(a) - \phi(a)(\omega)h$ for a in A. Since ψ is selfadjoint and $\psi(a)(\omega) = 0$ for all a in A, the argument in (1) implies that there exist positive linear maps ψ^+, ψ^-: $A \to C(T)$ such that $\psi = \psi^+ - \psi^-$. Moreover we have positive linear functionals η^+, η^- on A such that $\phi(a)(\omega) = \eta^+(a) - \eta^-(a)$ for all a in A. Define positive linear maps ϕ^+, ϕ^-: $A \to C(T)$ by $\phi^+(a) = \psi^+(a) + \eta^+(a)h$ and $\phi^-(a) = \psi^-(a) + \eta^-(a)h$. Then $\phi = \phi^+ - \phi^-$. This completes the proof.

REMARKS. (i) In [**11**] the following conjecture was made: Given a unital C^*-algebra A, if every completely bounded map of any C^*-algebra into A is decomposed as a linear combination of completely positive maps, then A is an injective C^*-algebra. In the case where A is commutative, the same conjecture was made in [**12**, pp. 97–98] as every bounded (resp. positive) linear map of a C^*-algebra into a commutative C^*-algebra is completed bounded (resp. positive) [**5**, Lemma 1; **9**, Chapter IV, Corollary 3.5]. Hence Theorem 1 may be regarded as a counterexample to the above conjecture of Tomiyama and Tsui.

(i) Let X be a compact Hausdorff space and assume that there exists a collection $\{X_i\colon i = 1, \ldots, n\}$ of subsets such that each X_i is homeomorphic to a stonean space, the intersection of any pair of sets of the collection is finite and $X = \bigcup_{i=1}^n X_i$. Theorem 1 then remains true with appropriate modifications if we replace T by X.

If there exist infinite subsets, we have the following situation. The proof is based on ideas due to Tsui [**12**, 1.3.4] and Wickstead [**13**, Theorem 3.15].

Let N be the set of all positive integers. For each i in N let X_i be a compact Hausdorff space with a limit point x_i and put $Y_i = X_i - \{x_i\}$. We denote by X the one-point compactification of the topological sum $\sum_i X_i$ of the sequence $\{X_i\}$; we denote by Y the one-point compactification of the topological sum $\sum_i Y_i$ of the sequence $\{Y_i\}$.

THEOREM 2. *With the above notation, there exists a bounded linear map ϕ of $C(X)$ into $C(Y)$ which cannot be decomposed as a linear combination of positive linear maps.*

PROOF. Define ϕ: $C(X) \to C(Y)$ as follows: for each f in $C(X)$,

$$\phi(f)(y) = f(x_i) - f(y) \quad \text{for } y \text{ in } Y_i,$$
$$\phi(f)(y_\infty) = 0$$

where y_∞ denotes the point at infinity. To see that $\phi(f)$ is continuous on Y, it suffices to prove that $\phi(f)$ is continuous at y_∞.

Let K be a compact subset of $\sum_i X_i$. Since each X_i is an open subset of $\sum_i X_i$, there exists m in N such that $K \subseteq \bigcup_{i=1}^m X_i$. Let x_∞ denote the point at infinity in X. Each neighborhood of x_∞ contains $X - K$ with such a compact subset K. Let a positive number r be given. Then there exists n in N such that $|f(y) - f(x_\infty)| < r/2$ for all y in $\bigcup_{i>n} X_i$. If $y \in Y_i$ with $i > n$,

$$|\phi(f)(y)| = |f(x_i) - f(y)| \leq |f(x_i) - f(x_\infty)| + |f(x_\infty) - f(y)| < r.$$

For each i in N with $i \leq n$, put $K_i = \{y \in X_i\colon\ |f(x_i) - f(y)| \geq r\}$. Then each K_i is a compact subset of Y_i. We may regard $K_1 \cup \cdots \cup K_n$ as a compact subset of $\sum_i Y_i$. Hence for y in $\sum_i Y_i - (K_1 \cup \cdots \cup K_n)$, we have $|\phi(f)(y)| < r$, so that $\phi(f)$ is continuous at y_∞. We have easily that ϕ is selfadjoint and $\|\phi\| = 2$.

Suppose that there exists a positive linear map ψ: $C(X) \to C(Y)$ such that $\psi \geq \phi$. For $g \geq 0$ in $C(X)$ and y in Y_i,

$$\psi(g)(y) \geq \phi(g)(y) = g(x_i) - g(y).$$

For every $y \in Y_i$, we can choose a continuous function h on X, such that $0 \leq h \leq 1$, $h(x_i) = 1$ and $h(y) = 0$. Hence for $g \geq 0$ in $C(X)$,

$$\psi(g)(y) \geq \psi(hg)(y) \geq h(x_i)g(x_i) - h(y)g(y) = g(x_i).$$

If a net $\{y_\alpha\}$ in Y_i converges to x_i, then $\{y_\alpha\}$, as a net in Y, converges to y_∞. Therefore we also have

$$\psi(g)(y_\infty) \geq g(x_i).$$

Let $k \in N$, and let $h_1, \ldots, h_k$ be continuous functions on X, such that $0 \leq h_i \leq 1$, $h_i(x_i) = 1$, $h_i(x) = 0$ for $x \in X - X_i$. Then

$$\psi(1)(y_\infty) \geq \psi\left(\sum_{i=1}^k h_i\right)(y_\infty) = \sum_{i=1}^k \psi(h_i)(y_\infty) \geq \sum_{i=1}^k h_i(x_i) = k.$$

This shows the unboundedness of ψ.

We now given a counterexample to the following conjecture of Smith [**8**, p. 165]: If B is a noninjective C^*-algebra such that $\sup\dim H_\pi = \infty$, where the supremum is taken over all irreducible representations π of B on H_π, then the norm closure of the linear span of completely positive maps of an infinite-dimensional C^*-algebra A into B is nowhere dense in the norm closure of the set of all completely bounded maps of A into B.

EXAMPLE 3. Let $L(H)$ be the C^*-algebra of all bounded linear operators on an infinite-dimensional Hilbert space H. With the space T as in Theorem 1, let $B = L(H) \oplus C(T)$, the direct sum of $L(H)$ and $C(T)$. Then B is a noninjective C^*-algebra such that every selfadjoint, completely bounded linear map ϕ of a unital C^*-algebra A into B is decomposed as a linear combination of completely positive maps and there exists an irreducible representation π of B on H.

PROOF. Since $C(T)$ is a noninjective C^*-algebra, so is B. There exist selfadjoint, completely bounded linear maps ϕ_1: $A \to L(H)$, ϕ_2: $A \to C(T)$ such that $\phi = \phi_1 \oplus \phi_2$. It follows from [**6**, Corollary 2.6] that ϕ_1 is decomposed as the difference of two completely positive maps of A into $L(H)$. Theorem 1 implies that ϕ_2 is decomposed as the difference of two completely positive maps of A into $C(T)$. Hence ϕ is decomposed as a linear combination of completely positive maps of A into B. The desired representation π is given by $\pi(a \oplus f) = a$ for $a \oplus f$ in $L(H) \oplus C(T)$.

3. Maps between partially ordered Banach spaces. We briefly discuss the positive decomposition of the Banach space of all bounded linear maps of a partially ordered Banach space into the Banach space of all continuous real functions on a compact Hausdorff space. Let X be a partially ordered Banach space with positive cone X_+. Then X_+ is normal if and only if there exists a constant $M > 0$ such that $0 \leq x \leq y$ implies that $M\|x\| \leq \|y\|$ [**7**, Chapter 2, Proposition 1.7]. Every bounded linear functional on X is the difference of two positive bounded linear functionals [**7**, Chapter 2, Proposition 1.21]. It is known that if S is stonean, then $L(X, C_r(S))$, the Banach space of all bounded linear maps of X into $C_r(S)$, is positively generated, that is, $L(X, C_r(S)) = L(X, C_r(S))_+ - L(X, C_r(S))_+$. Wickstead [**13**, Theorem 3.15] showed that given a compact metric space S, if $L(X, C_r(S))$ is positively generated whenever X_+ is normal, then S is stonean (finite). In the absence of the separability assumption, we have the following result.

THEOREM 4. *With the nonstonean compact Hausdorff space T as in Theorem* 1, *if X is a partially ordered Banach space with normal positive cone, then $L(X, C_r(T))$ is positively generated.*

PROOF. Since every bounded linear map of X into $C_r(S_i)$ $(i = 1, 2)$ is decomposed as the difference of two positive bounded linear maps, we can repeat the proof of Theorem 1.

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