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INVARIANTS RELATED TO THE BERGMAN KERNEL OF A BOUNDED DOMAIN IN $\mathbf{C}^n$

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ABSTRACT. In this paper we introduce biholomorphic invariants using the Bergman kernel function of a bounded domain in $\mathbf{C}^n$.

Let $K_D(z,\bar{t})$ be the Bergman kernel function of a bounded domain D in $\mathbf{C}^n$. As is well known, $K_D(z,\bar{t})$ admits the following transformation rule [**1**]:

Let D, Δ be bounded domains and $w = w(z)$ a biholomorphic mapping from D onto Δ. Then

$$K_D(z,\bar{t}) = \overline{\det \frac{d\tau}{dt}} K_\Delta(w,\bar{\tau}) \det \frac{dw}{dz} \qquad (\tau = w(t)). \tag{1}$$

Moreover,

$$T_D(z,\bar{t}) = \frac{\partial^2}{\partial t^* \, \partial z} \log K_D(z,\bar{t}),$$

which is defined when $K_D(z,\bar{t}) \neq 0$ and is uniquely determined by D, is a relative invariant under biholomorphic mappings, that is,

$$T_D(z,\bar{t}) = \left(\frac{d\tau}{dt}\right)^* T_\Delta(w,\bar{\tau}) \frac{dw}{dz}. \tag{2}$$

In particular, the Bergman metric

$$ds^2 = dz^* T_D(z,\bar{z})\, dz$$

is invariant under biholomorphic mappings.

Throughout this paper we use the following notation: $z = (z_1, z_2, \ldots, z_n)'$, $w = w(z) = (w_1(z), w_2(z), \ldots, w_n(z))'$,

$$\frac{\partial}{\partial z} = \left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_n}\right), \quad \frac{dw}{dz} = \frac{\partial w}{\partial z} = \frac{\partial}{\partial z} \times w,$$

where the symbols $'$, * and $\times$ stand for transposition, conjugated transposition and Kronecker product, respectively.

The above invariants make it possible to introduce some other biholomorphic invariants.

We define

$$K_{D,(p,q)}(z,\bar{t}) = K_D^p(z,\bar{t})(\det T_D(z,\bar{t}))^q \qquad (p, q \geq 0),$$

$$T_{D,(p,q)}(z,\bar{t}) = \frac{\partial^2}{\partial t^* \partial z} \log K_{D,(p,q)}(z,\bar{t}).$$

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Then we have the following formulas [**3**]:

$$K_{D,(p,q)}(z,\bar{t}) = \overline{\left(\det \frac{d\tau}{dt}\right)^{p+q}} K_{\Delta,(p,q)}(w,\bar{\tau}) \left(\det \frac{dw}{dz}\right)^{p+q}, \tag{3}$$

$$T_{D,(p,q)}(z,\bar{t}) = \left(\frac{d\tau}{dt}\right)^{*} T_{\Delta,(p,q)}(w,\bar{\tau}) \left(\frac{dw}{dz}\right). \tag{4}$$

In particular,

$$d\sigma_D^2 = ds^2_{D,(p,q)} = dz^* T_{D,(p,q)}(z,\bar{z})\, dz$$

is a Kähler metric which is invariant under biholomorphic mappings.

We note that our metric $d\sigma^2$ gives the Bergman metric for $p = 1,\ q = 0$, and the Burbea metric for $p = n+1,\ q = 1$ [**2**, **3**].

Making use of (1) and (4), it can be shown that

$$J_{D,(p,q)}(z,\bar{z}) = \frac{\det T_{D,(p,q)}(z,\bar{z})}{K_D(z,\bar{z})}$$

is a positive biholomorphic invariant.

Similarly, from (3) we also deduce that

$$H_{D,(p,q)}(z,\bar{t}) = \frac{K_{D,(p,q)}(z,\bar{t}) K_{D,(p,q)}(t,\bar{z})}{K_{D,(p,q)}(t,\bar{t}) K_{D,(p,q)}(z,\bar{z})}$$

is a positive biholomorphic invariant. This extends the result in [**5**] for the special case of $p = 1$ and $q = 0$.

We shall now define $R(z,\bar{t})$ by

$$R(z,\bar{t}) = \sqrt{\det T_{D,(p,q)}(t,\bar{t})}\, \left|\det SU(z,\bar{t})\right|,$$

where

$$U(z,\bar{t}) = T^{-1}_{D,(p,q)}(t,\bar{t}) \int_t^z T_{D,(p,q)}(z,\bar{t})\, dz,$$

and S is a vector differential operator such that

$$S = \left(\frac{d}{d\sigma}, \frac{d^2}{d\sigma^2}, \ldots, \frac{d^n}{d\sigma^n}\right), \quad d\sigma = ds_{D,(p,q)} = \sqrt{dz^* T_{D,(p,q)}(z,\bar{z})\, dz}.$$

Then we have the following

THEOREM. *$R(z,\bar{t})$ is a nonnegative biholomorphic invariant.*

PROOF. Let D, Δ be bounded domains and $w = w(z)$ be a biholomorphic mapping from D onto Δ. Then from (4) we have

$$\begin{aligned} U(z,\bar{t}) &= T^{-1}_{D,(p,q)}(t,\bar{t}) \int_t^z T_{D,(p,q)}(z,\bar{t})\, dz \\ &= \left(\frac{d\tau}{dt}\right)^{-1} T^{-1}_{\Delta,(p,q)}(\tau,\bar{\tau}) \left(\frac{d\tau}{dt}\right)^{*-1} \int_\tau^w \left(\frac{d\tau}{dt}\right)^{*} T_{\Delta,(p,q)}(w,\bar{\tau}) \frac{dw}{dz} \cdot \frac{dz}{dw}\, dw \\ &= \left(\frac{d\tau}{dt}\right)^{-1} T^{-1}_{\Delta,(p,q)}(\tau,\bar{\tau}) \int_\tau^w T_{\Delta,(p,q)}(w,\bar{\tau})\, dw, \end{aligned}$$

where $\tau = w(t)$. Thus

$$U(z,\bar{t}) = (d\tau/dt)^{-1}U(w,\bar{\tau}).$$

Differentiating this equation with respect to the metric $d\sigma$, we obtain

$$U^{(n)}(z,\bar{t}) \equiv \frac{d^n}{d\sigma^n}U(z,\bar{t}) = \left(\frac{d\tau}{dt}\right)^{-1} U^{(n)}(w,\bar{\tau}). \tag{5}$$

It follows from (4) and (5) that

$$\begin{aligned} R(z,\bar{t}) &= \sqrt{\det T_{D,(p,q)}(t,\bar{t})}\,|\det SU(z,\bar{t})| \\ &= \sqrt{\det\left(\left(\frac{d\tau}{dt}\right)^{*} T_{\Delta,(p,q)}(\tau,\bar{\tau}) \left(\frac{d\tau}{dt}\right)\right)}\left|\det\left(\left(\frac{d\tau}{dt}\right)^{-1} SU(w,\bar{\tau})\right)\right| \\ &= \sqrt{\det T_{\Delta,(p,q)}(\tau,\bar{\tau})}\,|\det SU(w,\bar{\tau})| \\ &= R(w,\bar{\tau}). \end{aligned}$$

This concludes the proof.

REMARK. This result agrees with that in [**4**] when $p = 1$ and $q = 0$.

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