IN cvARIALTS RELATED TO THE BERGMAN KERNEL
OF A BOUNDED DOMAIN IN C^n
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ABSTRACT. In this paper we introduce biholomorphic invariants using the
Bergman kernel function of a bounded domain in C^n.

Let $K_D(z, \bar{t})$ be the Bergman kernel function of a bounded domain $D$ in $C^n$. As
is well known, $K_D(z, \bar{t})$ admits the following transformation rule [1]:

Let $D, \Delta$ be bounded domains and $w = w(z)$ a biholomorphic mapping from $D$
onto $\Delta$. Then

\[
K_D(z, \bar{t}) = \det \frac{d\tau}{dt} K_\Delta(w, \bar{\tau}) \det \frac{dw}{dz} \quad (\tau = w(t)).
\]

Moreover,

\[
T_D(z, \bar{t}) = \frac{\partial^2}{\partial t^* \partial z} \log K_D(z, \bar{t}),
\]

which is defined when $K_D(z, \bar{t}) \neq 0$ and is uniquely determined by $D$, is a relative
invariant under biholomorphic mappings, that is,

\[
T_D(z, \bar{t}) = \left( \frac{dT}{dt} \right)^* T_\Delta(w, \bar{\tau}) \frac{dw}{dz}.
\]

In particular, the Bergman metric

\[
ds^2 = dz^* T_D(z, \bar{z}) dz
\]
is invariant under biholomorphic mappings.

Throughout this paper we use the following notation: $z = (z_1, z_2, \ldots, z_n)'$, $w = w(z) = (w_1(z), w_2(z), \ldots, w_n(z))'$,

\[
\frac{\partial}{\partial z} = \left( \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_n} \right), \quad \frac{dw}{dz} = \frac{\partial w}{\partial z} = \frac{\partial}{\partial z} \times w,
\]

where the symbols $'$, $^*$ and $\times$ stand for transposition, conjugated transposition and
Kronecker product, respectively.

The above invariants make it possible to introduce some other biholomorphic
invariants.

We define

\[
K_{D,(p,q)}(z, \bar{t}) = K_D^p(z, \bar{t})(\det T_D(z, \bar{t}))^q \quad (p, q \geq 0),
\]

\[
T_{D,(p,q)}(z, \bar{t}) = \frac{\partial^2}{\partial t^* \partial z} \log K_{D,(p,q)}(z, \bar{t}).
\]
Then we have the following formulas [3]:

\[ K_{D,(p,q)}(z,\bar{t}) = \left( \det \frac{d\tau}{dt} \right)^{p+q} K_{\Delta,(p,q)}(w,\bar{\tau}) \left( \det \frac{dw}{dz} \right)^{p+q}, \]

\[ T_{D,(p,q)}(z,\bar{t}) = \left( \frac{d\tau}{dt} \right)^* T_{\Delta,(p,q)}(w,\bar{\tau}) \left( \frac{dw}{dz} \right). \]

In particular,

\[ d\sigma^2_D = ds^2_{D,(p,q)} = dz^* T_{D,(p,q)}(z,\bar{z}) \, dz \]

is a Kähler metric which is invariant under biholomorphic mappings.

We note that our metric \( d\sigma^2 \) gives the Bergman metric for \( p = 1, q = 0 \), and the Burbea metric for \( p = n + 1, q = 1 \) [2, 3].

Making use of (1) and (4), it can be shown that

\[ J_{D,(p,q)}(z,\bar{z}) = \frac{\det T_{D,(p,q)}(z,\bar{z})}{K_D(z,\bar{z})} \]

is a positive biholomorphic invariant.

Similarly, from (3) we also deduce that

\[ H_{D,(p,q)}(z,\bar{t}) = \frac{K_{D,(p,q)}(z,\bar{t})K_{D,(p,q)}(t,\bar{z})}{K_{D,(p,q)}(t,\bar{t})K_{D,(p,q)}(z,\bar{z})} \]

is a positive biholomorphic invariant. This extends the result in [5] for the special case of \( p = 1 \) and \( q = 0 \).

We shall now define \( R(z,\bar{t}) \) by

\[ R(z,\bar{t}) = \sqrt{\det T_{D,(p,q)}(t,\bar{t})} \left| \det S U(z,\bar{t}) \right|, \]

where

\[ U(z,\bar{t}) = T_{D,(p,q)}^{-1}(t,\bar{t}) \int_t^z T_{D,(p,q)}(z,\bar{t}) \, dz, \]

and \( S \) is a vector differential operator such that

\[ S = \left( \frac{d}{d\sigma}, \frac{d^2}{d\sigma^2}, \ldots, \frac{d^n}{d\sigma^n} \right), \quad d\sigma = ds_{D,(p,q)} = \sqrt{dz^* T_{D,(p,q)}(z,\bar{z}) \, dz}. \]

Then we have the following

**THEOREM.** \( R(z,\bar{t}) \) is a nonnegative biholomorphic invariant.

**PROOF.** Let \( D, \Delta \) be bounded domains and \( w = w(z) \) be a biholomorphic mapping from \( D \) onto \( \Delta \). Then from (4) we have

\[ U(z,\bar{t}) = T_{D,(p,q)}^{-1}(t,\bar{t}) \int_t^z T_{D,(p,q)}(z,\bar{t}) \, dz \]

\[ = \left( \frac{d\tau}{dt} \right)^{-1} T_{\Delta,(p,q)}^{-1}(\tau,\bar{\tau}) \left( \frac{d\tau}{dt} \right)^{-1} \int_\tau^w \left( \frac{d\tau}{dt} \right)^* T_{\Delta,(p,q)}(w,\bar{\tau}) \frac{dw}{dz} \cdot \frac{dz}{dw} \]

\[ = \left( \frac{d\tau}{dt} \right)^{-1} T_{\Delta,(p,q)}^{-1}(\tau,\bar{\tau}) \int_\tau^w T_{\Delta,(p,q)}(w,\bar{\tau}) \, dw, \]
where \( \tau = w(t) \). Thus
\[
U(z, \bar{t}) = \frac{dt}{d\tau}^{-1} U(w, \bar{\tau}).
\]
Differentiating this equation with respect to the metric \( d\sigma \), we obtain
\[
U^{(n)}(z, \bar{t}) \equiv \frac{d^n}{d\sigma^n} U(z, \bar{t}) = \left( \frac{dt}{d\tau} \right)^{-1} U^{(n)}(w, \bar{\tau}).
\]
It follows from (4) and (5) that
\[
R(z, \bar{t}) = \sqrt{\frac{\det T_D,(p,q)(t, \bar{t})}{\det SU(z, \bar{t})}} \frac{\det SU(w, \bar{\tau})}{\det T_D,(p,q)(\tau, \bar{\tau})} \frac{dt}{d\tau}^{-1}.
\]
This concludes the proof.

Remark. This result agrees with that in [4] when \( p = 1 \) and \( q = 0 \).

References

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