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ON THE INJECTIVITY OF THE ATTENUATED RADON TRANSFORM

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ABSTRACT. We show that the attenuated (exponential) Radon transform R_μ, where μ is assumed to be linear in the space variable, is injective on compactly supported distributions. Moreover, a limited angle reconstruction is possible and a hole theorem holds. We review the well-known special case of constant attenuation.

Introduction. Let $\mu \in C^\infty(S^1 \times \mathbf{R}^2)$ be given. The *attenuated Radon transform* is the operator R_μ: $C_c^\infty(\mathbf{R}^2) \to C_c^\infty(S^1 \times \mathbf{R})$, defined by

$$R_\mu \varphi(\omega, p) = \int_{\omega \cdot x = p} \varphi(x) e^{\mu(\omega, x)}\, dx, \tag{0.1}$$

where dx denotes the Lebesgue measure on the line $\omega \cdot x = p$. The adjoint R_μ^t: $C^\infty(S^1 \times \mathbf{R}) \to C^\infty(\mathbf{R}^2)$ of R_μ is given by

$$R_\mu^t \psi(x) = \int_{S^1} \psi(\omega, \omega \cdot x) e^{\mu(\omega, x)}\, d\omega.$$

As in the classical case $\mu \equiv 0$ (cf. [**2**]), we extend R_μ from $\mathcal{E}'(\mathbf{R}^2)$ into $\mathcal{E}'(S^1 \times \mathbf{R})$ via $\langle R_\mu u, \psi \rangle = \langle u, R_\mu^t \psi \rangle$, where $u \in \mathcal{E}'(\mathbf{R}^2)$ and $\psi \in C^\infty(S^1 \times \mathbf{R})$.

The case $\mu(\omega, x) = (\text{const})\omega^\perp \cdot x$, where $\omega^\perp = (-\omega_2, \omega_1)$, is known as *constant attenuation* (see e.g. Tretiak and Metz [**9**]). In this note we assume throughout that μ is *linear* in x, i.e. there exists a C^∞ vector field $\nu = (\nu_1, \nu_2)$: $S^1 \to \mathbf{R}^2$, such that

$$\mu(\omega, x) = x \cdot \nu(\omega) = x_1 \nu_1(\theta) + x_2 \nu_2(\theta), \tag{0.2}$$

where $\theta \in [0, 2\pi]$ and $\omega = (\cos\theta, \sin\theta)$. Thus (0.2) can be regarded as a first order approximation to a general $\mu \in C^\infty$.

The results of this paper (except for the limited angle reconstruction) are known for constant attenuation. Tretiak and Metz [**9**] and Markoe [**3**] gave inversion formulae, and Quinto [**6**] proved the hole theorem in this case (his proof is not related to ours). Further, Quinto [**6**] inverted rotation invariant Radon transforms. But (0.1) is *not* rotation invariant, and in such cases only local invertibility results are known so far (Markoe and Quinto [**4**]).

Transforms of type (0.1) appear in problems pertaining to emission computed tomography, see e.g. Tretiak [**8**]. Of course, our injectivity result (§1) only demonstrates that a subclass (0.2) of such transforms can *in principle* be inverted. The limited angle theorem and the hole theorem (§2) are at least of theoretical interest (they are of even practical interest for the classical Radon transform).

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1. Injectivity and limited angle reconstruction. Let $u \in \mathcal{E}'(\mathbf{R}^2)$ and μ as in (0.2). Then for $\omega \in S^1$ and $p \in \mathbf{R}$

$$\hat{u}(p\omega + i\nu(\omega)) = (\text{const})(R_\mu u)\hat{\ }(\omega, p), \tag{1.1}$$

where $\hat{u}$ is the Fourier transform of u and $(R_\mu u)\hat{\ }$ the Fourier transform of $R_\mu u$ w.r.t. the second variable. This *projection theorem* is verified as in the classical or constantly attenuated case (cf. Natterer [**5**]). Now let I be a nonvoid open interval in $[0, 2\pi]$. Because of relation (1.1) we are interested in the surface

$$M_\mu = \{z = p\omega + i\nu(\omega)\colon\ p \in \mathbf{R},\ \theta \in I\} \subset \mathbf{C}^2, \tag{1.2}$$

where again $\omega = (\cos\theta, \sin\theta)$. For $z \in M_\mu$ let $T_z M_\mu$ be the real-linear span of $\partial z/\partial p$ and $\partial z/\partial\theta$:

$$T_z M_\mu = \langle \partial z/\partial p, \partial z/\partial\theta\rangle.$$

Since $\hat{u}$ is entire, by (1.1) injectivity of R_μ means essentially the existence of points z in M_μ without complex tangents in $T_z M_\mu$, i.e. the complexification $T_z M_\mu \otimes \mathbf{C}$ of $T_z M_\mu$ is whole of $\mathbf{C}^2$. In particular, at such points, $\partial z/\partial p$ and $\partial z/\partial\theta$ form a basis of $\mathbf{C}^2$ (cf. [**10**, §17]).

PROPOSITION 1.1. *M_μ is a set of uniqueness for entire functions on $\mathbf{C}^2$. Indeed, if $z \in M_\mu$ with $p \neq 0$ then $\dim(T_z M_\mu) = 2$ and*

$$T_z M_\mu \cap i T_z M_\mu = \{0\}, \tag{1.3}$$

i.e. the Taylor coefficients at z of an entire function can be reconstructed from its values on a neighborhood of z in M_μ.

PROOF. First note that

$$\partial z/\partial p = (\cos\theta, \sin\theta), \quad \partial z/\partial\theta = (-p\sin\theta, p\cos\theta) + i(\nu_1'(\theta), \nu_2'(\theta)).$$

Hence condition (1.3) gives rise to a homogeneous system of four linear equations, which matrix A is given by

$$A = \begin{pmatrix} 0 & -\nu_1'(\theta) & \cos\theta & -p\sin\theta \\ 0 & -\nu_2'(\theta) & \sin\theta & p\cos\theta \\ \cos\theta & -p\sin\theta & 0 & \nu_1'(\theta) \\ \sin\theta & p\cos\theta & 0 & \nu_2'(\theta) \end{pmatrix}.$$

The determinant of A is immediately computed to

$$\det(A) = p^2 + (\nu_2'\cos\theta - \nu_1'\sin\theta)^2,$$

which means $\det(A) \neq 0$ if $p \neq 0$. This proves $\dim(T_z M_\mu) = 2$ as well as equation (1.3) for $p \neq 0$. The rest of the proposition is now clear by the following remark. □

REMARK. How can an entire function f on $\mathbf{C}^2$ be recovered explicitly in the situation of Proposition 1.1?—To this purpose, let $z^0 \in M_\mu$ and assume we know f in a neighbourhood of z^0 on M_μ. Then we know its partial derivatives in the (complex) directions $v = \partial z/\partial p$ and $w = \partial z/\partial\theta$, which (by Proposition 1.1) form a basis of $\mathbf{C}^2$. Let A be a regular complex-linear map on $\mathbf{C}^2$, such that

$$Av = (1, 0) \quad \text{and} \quad Aw = (0, 1).$$

Applying Taylor's formula to the entire function $f \circ A^{-1}$, we obtain for $z \in \mathbf{C}^2$

$$f(A^{-1}z) = \sum_{\alpha+\beta=0}^{\infty} \frac{1}{\alpha!\beta!} \frac{\partial^{\alpha+\beta} f}{\partial^\alpha v \partial^\beta w}(z^0)(z_1 - a_1)^\alpha (z_2 - a_2)^\beta,$$

where $a = Az^0 \in \mathbf{C}^2$. Thus we know f everywhere on $\mathbf{C}^2$.

The following theorems are an easy consequence of Proposition 1.1.

THEOREM 1.2. *R_μ: $\mathcal{E}'(\mathbf{R}^2) \to \mathcal{E}'(S^1 \times \mathbf{R})$ is injective.*

THEOREM 1.3 (LIMITED ANGLE THEOREM). *If $u \in \mathcal{E}'(\mathbf{R}^2)$ then $R_\mu u$ is determined by its values on any set $\Omega \times \mathbf{R}$, where Ω is nonvoid and open in S^1.*

REMARK. For a submanifold M of $\mathbf{C}^2$ and its tangent space $T_z M$, condition (1.3) is known as M being *totally real* at z. In general we do not know that M_μ is a submanifold. But for constant attenuation a more precise description of M_μ is possible, see §3.

2. The hole theorem. Recall that μ is of the form (0.2), but for this section it suffices that ν: $S^1 \to \mathbf{R}^2$ is a C^1 vector field.—The proof of the following theorem is in its idea a variant of Helgason's proof for the classical case [**2**], of which a short presentation was given by Strichartz in [**7**].

THEOREM 2.1 (HOLE THEOREM). *Let φ be a Lipschitz continuous function of compact support on $\mathbf{R}^2$. Then for $r > 0$*

$$\operatorname{supp}(R_\mu \varphi) \subset \{|p| \le r\} \Rightarrow \operatorname{supp}(\varphi) \subset \{|x| \le r\},$$

where supp *denotes the supports of the respective functions.*

PROOF. We assume that $R_\mu \varphi$ is supported by $\{(\omega, p): |p| \le r\}$. It suffices to show that φ vanishes on every line $x_1 = p$ with $p > r$; since if U is a rotation, then

$$R_\mu \varphi(U\omega, p) = R_{\mu'} \varphi_U(\omega, p),$$

where $\mu'(\omega, x) = x \cdot U^{-1}(\nu(U\omega))$ and $\varphi_U(x) = \varphi(Ux)$, i.e. μ' is again of type (0.2). Thus it remains to prove

$$\int t\varphi(p,t)\exp(p\nu_1(0) + t\nu_2(0))\,dt = 0, \tag{2.1}$$

then (by induction) we can replace the factor t in (2.1) by any polynomial in p and t, and the Weierstrass theorem finally gives $\varphi(p, \cdot) \equiv 0$. To show (2.1) let $U_\theta(p,t) = (p\cos\theta + t\sin\theta, -p\sin\theta + t\cos\theta)$ be a rotation by the angle θ. By assumption we know

$$\int \varphi(U_\theta(p,t))\exp(U_\theta(p,t) \cdot \nu(e^{i\theta}))\,dt = 0 \tag{2.2}$$

for all θ. Differentiating (2.2) with respect to θ and putting $\theta = 0$ yields

$$\begin{aligned} &\int \varphi(p,t)\exp(p\nu_1(0) + t\nu_2(0))(p\nu_1'(0) + t\nu_2'(0) + t\nu_1(0) - p\nu_2(0))\,dt \\ &\qquad + \int (t\partial\varphi/\partial p - p\partial\varphi/\partial t)\exp(p\nu_1(0) + t\nu_2(0))\,dt = 0. \end{aligned} \tag{2.3}$$

Since φ is Lipschitz continuous, integration by parts shows

$$-\int \partial\varphi/\partial t \exp(p\nu_1(0)+t\nu_2(0))\,dt = 0,$$

using (2.2) with $\theta = 0$. Applying (2.2) also to the first integral in (2.3) we obtain therefore

$$\begin{aligned}\int \varphi(p,t)\exp(p\nu_1(0)+t\nu_2(0))t(\nu_2'(0)+\nu_1(0))\,dt \\ + \int t\partial\varphi/\partial p \exp(p\nu_1(0)+t\nu_2(0))\,dt = 0.\end{aligned} \tag{2.4}$$

We put $a = \nu_2'(0)+\nu_1(0)$ and divide by $\exp(p\nu_1(0))$ in (2.4). Thus

$$(a+\partial/\partial p)\int t\varphi(p,t)\exp(t\nu_2(0))\,dt \equiv 0,$$

for $p > r$. But this implies

$$\int t\varphi(p,t)\exp(t\nu_2(0))\,dt = (\mathrm{const})e^{-ap} \tag{2.5}$$

for all $p > r$. Since φ has compact support, the integral in (2.5) must vanish for large p, i.e. const $= 0$. This shows (2.1) and the theorem is proved. □

REMARK 2.2. Theorem 2.1 also implies the injectivity of R_μ on $C_c^1(\mathbf{R}^2)$. Our proof does not show directly whether the hole theorem holds on $\mathcal{E}'(\mathbf{R}^2)$ for R_μ (as in the classical [**2**] or rotation invariant [**6**] case).

3. The case of constant attenuation. We finally study M_μ (cf. §1) in the case of constant attenuation. A precise description of M_μ is possible then, and if $\mu \neq 0$ the location of M_μ in $\mathbf{C}^2$ shows an interesting difference to the classical situation $M_\mu = \mathbf{R}^2$. As usual, for $z = (z_1, z_2) \in \mathbf{C}^2$ we write $z_1 = x_1 + ix_3$ and $z_2 = x_2 + ix_4$.

THEOREM 3.1. *If $\mu(\omega, x) = cx\cdot\omega^\perp$, $c \neq 0$, then M_μ is an algebraic submanifold of $\mathbf{C}^2$, given by the equations*

$$x_3^2 + x_4^2 - c^2 = 0 \quad \text{and} \quad x_1x_3 + x_2x_4 = 0. \tag{3.1}$$

A point $z \in M_\mu$ is not totally real if and only if $x_1 = x_2 = 0$, i.e. if $p = 0$ in (1.2).

PROOF. The proof is straightforward if we observe that a point of a manifold $M = \{\rho_1(x) = \rho_2(x) = 0\}$ is totally real if and only if $\det(\partial\rho_m/\bar{\partial}z_j) \neq 0$ at this point, see Cirka [**1**]. In fact, in our case we have from (3.1) that $\det(\partial\rho_m/\bar{\partial}z_j) = cp/2$. □

In particular, if $c \neq 0$, M_μ contains points z at which *not all* Taylor coefficients of an entire function on $\mathbf{C}^2$ can be reconstructed (from a neighbourhood of z in M_μ).

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