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A WAVE EQUATION WITH A POSSIBLY JUMPING NONLINEARITY

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ABSTRACT. Existence of a doubly periodic solution to a forced semilinear wave equation is established. The nonlinearity may "jump" across any finite number of eigenvalues of finite multiplicity.

1. Introduction. Let $J = [0, 2\pi] \times [0, 2\pi]$ and let $f\colon J \times \mathbf{R} \to \mathbf{R}$, $(t, x, s) \mapsto f(t, x, s)$, be a function satisfying the Carathéodory conditions. Assume there is a number $A > 0$ and a function $B \in L^2(J)$ such that for each $s \in \mathbf{R}$ and $(t, x) \in J$ we have

$$|f(t, x, s)| \leq A|s| + B(t, x). \tag{1.1}$$

Let $h \in L^2(J)$. We consider the existence (in the weak sense) of solutions 2π-periodic in each of x and t for the semilinear wave equation

$$u_{tt} - u_{xx} - f(t, x, u) = h(t, x). \tag{1.2}$$

By a weak solution to the doubly 2π-periodic problem for (1.2) is meant a $u \in L^2(J)$ such that

$$\begin{aligned}\int_J u(t,x)[v_{tt}(t,x) - v_{xx}(t,x)]\,dt\,dx \\ = \int_J [f(t,x,u(t,x)) + h(t,x)]v(t,x)\,dt\,dx\end{aligned} \tag{1.3}$$

for every $v \in C^2(J)$ satisfying the boundary conditions

$$\begin{aligned} v(t,0) - v(t,2\pi) = v_x(t,0) - v_x(t,2\pi) = 0 \qquad (t \in [0,2\pi]),\\ v(0,x) - v(2\pi,x) = v_t(0,x) - v_t(2\pi,x) = 0 \qquad (x \in [0,2\pi]).\end{aligned}$$

If $\lambda \in \mathbf{R}$ the doubly 2π-periodic problem for

$$u_{tt} - u_{xx} - \lambda u = h(t, x) \tag{1.4}$$

has a unique weak solution for every $h \in L^2(J)$ if and only if $\lambda \notin \Sigma$, where

$$\Sigma = \{n^2 - m^2 : (m, n) \in Z \times Z\} = \{\ldots, \lambda_{-2}, \lambda_{-1}, \lambda_0 = 0, \lambda_1, \lambda_2, \ldots\}$$

and Z denotes the integers.

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Existence results for (1.2) with any of the usual boundary conditions (e.g., doubly periodic, periodic-Dirichlet) usually require f to be monotone in s. The monotonicity enables one to work around difficulties created by the infinite multiplicity of the eigenvalue $\lambda_0 = 0$.

It is known that, with our boundary conditions, (1.2) has a solution for each $h \in H$ if f is monotone in s and is asymptotically between (and bounded away from) two successive eigenvalues (Mawhin [**M.1**]). or if f is monotone in s and "jumps" (asymptotically, going from $-\infty$ to $+\infty$) from one eigenvalue to the next, or to the one below (provided neither is $\lambda_0 = 0$) (Willem [**Wi**]).

These results have been recently unified and generalized to include nonuniformities in the avoidance of Σ (Mawhin and Ward [**M-W.1**, **M-W.2**]). Here we show that f may jump across arbitrarily many eigenvalues of finite multiplicity and (1.3) may still be solvable for all $h \in H$.

2. Statement of results. The following is our main result.

THEOREM 1. *Let f: $J \times \mathbf{R} \to \mathbf{R}$ satisfy the Carathéodory conditions and* (1.1). *Let $h \in L^2(J)$ and suppose*:

(c1) *$f(t,x,s)$ is monotone nondecreasing in s for each $(t,x) \in J$.*

(c2) *There is a number $\alpha > 0$ and a function $\beta \in H$ such that*

$$|f(t,x,s)| \le f(t,x,s) + \alpha|s| + \beta(t,x)$$

for all $(t,x,s) \in J \times \mathbf{R}$.

(c3) *There is a number $\eta_0 > 0$ such that $0 < \eta_0 \le \underline{\lim}_{|s|\to\infty} s^{-1} f(t,x,s)$, uniformly in $(t,x) \in J$.*

Then there is a number $\alpha_0 > 0$ such that a weak solution to the doubly 2π-periodic problem for (1.2) *exists whenever $\alpha < \alpha_0$.*

REMARK 1. One may instead assume $f(t,x,s)$ is nonincreasing in s and the existence of $\eta_0 < 0$ with $0 > \eta_0 \ge \overline{\lim}_{|s|\to\infty} s^{-1} f(t,x,s)$. One may also replace f in (c2) by $-f$.

As a corollary we have the following result on jumping nonlinearities. Consider the equation

$$u_{tt} - u_{xx} + \alpha_- u^- - \alpha_+ u^+ - g(t,x,u) = h(t,x) \tag{2.1}$$

where $u^+ = \max(u,0)$, $u^- = \max(-u,0)$, and $u = u^+ - u^-$. Suppose $h \in L^2(J)$ and g: $J \times \mathbf{R} \to \mathbf{R}$ satisfies the Carthéodory conditions and (1.1).

COROLLARY 1. *Let α_- and α_+ be positive numbers. Suppose $f(t,x,s) := -\alpha_- s^- + \alpha_+ s^+ + g(t,x,s)$ is monotone nondecreasing in s and*

$$\lim_{|s|\to\infty} s^{-1} g(t,x,s) = 0,$$

uniformly for $(t,x) \in s$. Then there is a number $\alpha_0 > 0$ such that (2.1) *has a weak doubly 2π-periodic solution provided $0 < \alpha_- < \alpha_0$ (or $0 < \alpha_+ < \alpha_0$). (α_0 does not depend on α_+, g, or h.)*

REMARK 2. A similar corollary is true with α_-, α_+ negative and $f(t,x,s)$ nonincreasing in s.

REMARK 3. We note that if $\alpha_- < \alpha_+$ then we may have $]\alpha_-, \alpha_+[\cap \Sigma \ne \emptyset$. Indeed, the interval $]\alpha_-, \alpha_+[$ may contain any finite number of positive eigenvalues.

REMARK 4. The corollary may be viewed as a surjectivity result concerning operators of the form $Lu + \alpha_- u^- - \alpha_+ u^+$, where L is the D'Alembertian realized in $L^2(J)$ with our boundary conditions.

3. Abstract formulation and proofs. Let $H = L^2(J)$ with inner product $(u, v) = \int_J uv\,dt\,dx$ and corresponding norm $\|\cdot\|$. Each $u \in H$ has a representation as a Fourier series of the form

$$u = \sum_{(m,n)\in Z\times Z} \alpha_{mn} u_{mn},$$

where $u_{mn} = e^{i(mt+nx)}$ for $(m, n) \in Z \times Z$ and $\alpha_{m,n} \in \mathbf{C}$ with $\overline{\alpha}_{m,n} = \alpha_{-m,-n}$ to make the sum real.

Let

$$D(L) = \left\{ u \in H : \sum_{(m,n)\in Z^2} (n^2 - m^2)^2 |\alpha_{mn}|^2 < \infty \right\}.$$

Define $L : D(L) \subseteq H \to H$ by

$$Lu = \sum_{(m,n)\in Z\times Z} (n^2 - m^2)\alpha_{mn} u_{mn} \quad \text{for } u \in D(L).$$

Let $F : H \to H$ be the substitution operator defined by f. It is known that u is a weak solution of the doubly 2π-periodic problem for (1.2) if and only if $u \in D(L)$ and

$$Lu - F(u) = h \tag{3.1}$$

PROOF OF THEOREM 1. Let λ_1 be the first positive eigenvalue of L (i.e., smallest positive number in Σ so that $\lambda_1 = 1$). Choose $\varepsilon_0 > 0$ so that $\varepsilon_0 < \min(\lambda_1, \eta_0)$. For $\lambda \in]0, 1[$ consider the family of equations

$$Lu - (1-\lambda)\varepsilon_0 u - \lambda F(u) = \lambda h. \tag{3.2}$$

The operator $(L - \varepsilon_0 I)^{-1}$ is not compact. Nevertheless, it follows from a theorem in Willem's paper [**Wi**] (or see [**M.2**]) that it suffices to show that all possible solutions of (3.2) are bounded in H independently of $\lambda \in]0, 1[$.

If (u, λ) is a solution of (3.2) with $0 < \lambda < 1$ then

$$Lu - (1-\lambda)\varepsilon_0 u - \lambda f(t, x, u) = \lambda h(t, x), \tag{3.3}$$

and by taking inner products with 1 we derive, since $(Lu, 1) = 0$,

$$\lambda \int_J f(t, x, u)\,dt\,dx = -(1-\lambda)\varepsilon_0 \int_J u\,dt\,dx - \lambda \int_J h(t, x)\,dt\,dx. \tag{3.4}$$

Taking absolute values in (3.3) and using (c2) we have a.e. on J:

$$\begin{aligned} |Lu(t,x)| \leq &(1-\lambda)\varepsilon_0 |u(t,x)| + \lambda f(t, x, u(t,x)) \\ &+ \alpha|u(t,x)| + \beta(t,x) + |h(t,x)|. \end{aligned}$$

Integrating over J and using (3.4) we obtain

$$|Lu|_{L^1} \leq (2\varepsilon_0 + \alpha)|u|_{L^1} + C_1 \tag{3.5}$$

where C_1 is a constant.

For $u \in H$ let us write $u = u_0 + u_1$ with $u_0 \in \ker L$ and $u_1 \in \ker L^{\perp} = \text{Range } L$. It is known (cf., e.g., [**L** or **C-H**]) that there is a constant $\mu > 0$ such that for $u = u_0 + u_1 \in D(L)$,

$$|u_1|_{L^\infty} \leq \mu |Lu_1|_{L^1}.$$

Thus for any solution u of (3.2) we have

$$|u_1|_{L^\infty} \leq \mu |Lu_1|_{L^1} \leq \mu(2\varepsilon_0 + \alpha)|u|_{L^1} + \mu C_1$$

and

$$|u_1|_{L^\infty} \leq (2\varepsilon_0 + \alpha) C_2 \|u\| + C_3 \tag{3.6}$$

for some constants C_2 and C_3. Of course, by (3.5) we also have

$$|Lu_1|_{L^1} \leq (2\varepsilon_0 + \alpha) C_4 \|u\| + C_1. \tag{3.7}$$

Taking the inner product of the expression on each side of (3.2) with u we derive

$$(1-\lambda)\varepsilon_0 \|u\|^2 + \lambda \int_J f(t, x, u) u = (Lu, u) - \lambda(h, u). \tag{3.8}$$

By condition (c3) there is a number $r > 0$ such that

$$f(t, x, s)s \geq \varepsilon_0 s^2$$

for $|s| \geq r$. Thus there is a function $\gamma \in H$ with

$$f(t, x, s)s \geq \varepsilon_0 s^2 - \gamma(t, x)$$

for all $(t, x, s) \in J \times \mathbf{R}$. From (3.8) we see that

$$(1-\lambda)\varepsilon_0 \|u\|^2 + \lambda \int_J \varepsilon_0 |u|^2 \, dt\, dx - \int_J \gamma \, dt\, dx \leq |(Lu, u)| + \|h\| \cdot \|u\|.$$

Thus there is a constant C with

$$\begin{aligned} \varepsilon_0 \|u\|^2 &\leq |(Lu_1, u_1)| + \|h\| \cdot \|u\| + C \\ &\leq |Lu_1|_{L^1} \cdot |u_1|_{L^\infty} + \|h\| \cdot \|u\| + C. \end{aligned}$$

By (3.6) and (3.7) we now obtain

$$\varepsilon_0 \|u\|^2 \leq [(2\varepsilon_0 + \alpha) C_4 \|u\| + C_1][(2\varepsilon_0 + \alpha) C_2 \|u\| + C_3] + \|h\| \cdot \|u\| + C$$

and, hence,

$$\varepsilon_0 \|u\|^2 \leq (2\varepsilon_0 + \alpha)^2 k_1 \|u\|^2 + k_2 \|u\| + k_3$$

for some constants k_1, k_2, and k_3.

A subtraction yields

$$[\varepsilon_0 - (2\varepsilon_0 + \alpha)^2 k_1] \|u\|^2 \leq k_2 \|u\| + k_3. \tag{3.9}$$

By now choosing ε_0 and α_0 sufficiently small we can insure that, since $\alpha < \alpha_0$,

$$\varepsilon_0 - (2\varepsilon_0 + \alpha)^2 k_1 > 0,$$

which, by (3.9), implies $\|u\| < M$ for some constant $M > 0$. All possible solutions of (3.2) are thus bounded independently of $\lambda \in]0, 1[$, and (3.1) has a solution.

PROOF OF THE COROLLARY. We take

$$f(t, x, s) = -\alpha_- s^- + \alpha_+ s^+ + g(t, x, s).$$

By hypothesis f is monotone nondecreasing in s. Also

$$\begin{aligned}|f(t,x,s)| &\leq \alpha_- s^- + \alpha_+ s^+ + |g(t,x,s)| \\ &\leq f(t,x,s) + 2\alpha_- s^- + 2|g(t,x,s)|.\end{aligned}$$

By hypothesis, for each $\varepsilon > 0$ there exists $\gamma_\varepsilon \in H$ with $|g(t,x,s)| \leq \varepsilon|s| + \gamma_\varepsilon(t,x)$. It follows that

$$|f(t,x,s)| \leq f(t,x,s) + (2\alpha_- + 2\varepsilon)|s| + 2\gamma_\varepsilon(t,x),$$

which shows (c2) holds. Since $\varepsilon > 0$ may be chosen arbitrarily small we can insure that $2\alpha_- + 2\varepsilon < \alpha_0$, where α_0 is the number in Theorem 1, by requiring $\alpha_- < \alpha_0/2$ and then choosing ε. Since

$$\underline{\lim}_{|s|\to\infty} s^{-1} f(t,x,s) \geq \min(\alpha_-, \alpha_+) > 0,$$

the corollary follows.

REMARK 5. Instead of looking for solutions 2π-periodic in t and x we could also formulate our results for solutions ω_1-periodic in t and ω_2-periodic in x if we insist that ω_1/ω_2 be a rational number. This would insure that the d'Alembertian with these boundary conditions is realized in H by a selfadjoint operator having properties like those of L above. If ω_1/ω_2 is irrational, small divisors appear in the right inverse of L which lead to unsolved difficulties.

4. A counterexample. It is easy to see that the corollary is false if $\alpha_- = 0$ and $\alpha_+ > 0$. For example, if $g \equiv 0$ we have

$$Lu - \alpha^+ u^+ = h, \tag{4.1}$$

and by taking inner products with 1 we see that h must satisfy $(h,1) \leq 0$. Thus (4.1) cannot be solvable for all $h \in H$. In spite of this, (4.1) is certainly solvable for some $h \in H$. One might expect a solution if

$$(h,1) = \int_J h\,dt\,dx < 0.$$

We show however that there may not be a solution even then.

It is easy to show that $u \in \ker L$ if and only if $u = p(t+x) + q(t-x)$ for some p, q each 2π-periodic on $\mathbf{R}$ with $p, q \in L^2(0, 2\pi)$.

Let $0 < \delta_1 < \delta_2 < \pi$ and $p\colon [0, 2\pi] \to R$ be at least C^2 smooth and defined by

$$p(s) = \begin{cases} 3 & \text{if } |\pi - s| \leq \delta_1, \\ 0 & \text{if } |\pi - s| \geq \delta_2 \text{ and } 0 \leq s \leq 2\pi, \\ 0 \leq p(s) \leq 3 & \text{elsewhere.} \end{cases}$$

Extend p 2π-periodically to all of $\mathbf{R}$ and define $\phi \in H$ by

$$\phi(t,x) = p(t+x) \quad \text{for } (t,x) \in J.$$

Then $\phi \in \ker L$; indeed, ϕ is a smooth (classical) solution of $u_{tt} - u_{xx} = 0$ and ϕ is 2π-periodic in each of x and t.

By choosing δ_2 sufficiently small we can insure that

$$\int_J \phi\,dt\,dx < 4\pi^2.$$

On the other hand, by choosing δ_1 sufficiently close to δ_2, one can insure that

$$\int_J \phi \, dt \, dx < \int_J \phi^2 \, dt \, dx.$$

Now consider (4.1) with $h = \phi - 1$. We observe that

$$\int_J h \, dt \, dx = \int_J \phi \, dt \, dx - 4\pi^2 < 0.$$

Suppose $u \in D(L)$ solves

$$Lu - \alpha_+ u^+ = \phi - 1. \tag{4.2}$$

Taking inner products with ϕ we obtain, since $(Lu, \phi) = 0$,

$$0 \geq -\alpha_+ \int_J u^+ \phi \, dt \, dx = \int_J (\phi - 1)\phi \, dt \, dx > 0.$$

Thus (4.2) cannot have a solution.

If we now let $h = \mu(\phi - 1)$ with $\mu \gg 1$ we see that there still is not a solution even when $-\int_J h$ is large.

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