A WAVE EQUATION
WITH A POSSIBLY JUMPING NONLINEARITY

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ABSTRACT. Existence of a doubly periodic solution to a forced semilinear
wave equation is established. The nonlinearity may "jump" across any finite
number of eigenvalues of finite multiplicity.

1. Introduction. Let \( J = [0, 2\pi] \times [0, 2\pi] \) and let \( f: J \times \mathbb{R} \to \mathbb{R}, \ (t, x, s) \mapsto f(t, x, s) \), be a function satisfying the Carathéodory conditions. Assume there is a
number \( A > 0 \) and a function \( B \in L^2(J) \) such that for each \( s \in \mathbb{R} \) and \((t, x) \in J\) we have

\[
|f(t, x, s)| \leq A|s| + B(t, x).
\]

Let \( h \in L^2(J) \). We consider the existence (in the weak sense) of solutions \( 2\pi \)-periodic in each of \( x \) and \( t \) for the semilinear wave equation

\[
utt - uxx - f(t, x, u) = h(t, x).
\]

By a weak solution to the doubly \( 2\pi \)-periodic problem for (1.2) is meant a \( u \in L^2(J) \) such that

\[
\int_J u(t, x)[vtt(t, x) - vxx(t, x)]
= \int_J [f(t, x, u(t, x)) + h(t, x)]v(t, x)
\]

for every \( v \in C^2(J) \) satisfying the boundary conditions

\[
v(t, 0) - v(t, 2\pi) = v_x(t, 0) - v_x(t, 2\pi) = 0 \quad (t \in [0, 2\pi]),
\]

\[
v(0, x) - v(2\pi, x) = v_x(0, x) - v_x(2\pi, x) = 0 \quad (x \in [0, 2\pi]).
\]

If \( \lambda \in \mathbb{R} \) the doubly \( 2\pi \)-periodic problem for

\[
utt - uxx - \lambda u = h(t, x)
\]

has a unique weak solution for every \( h \in L^2(J) \) if and only if \( \lambda \notin \Sigma \), where

\[
\Sigma = \{n^2 - m^2 : (m, n) \in Z \times Z\} = \{\ldots, -1, 0, 1, 2, \ldots\}
\]

and \( Z \) denotes the integers.

1. This work was done while the author was visiting the Université Catholique de Louvain,
Louvain-la-Neuve, Belgium.
Existence results for (1.2) with any of the usual boundary conditions (e.g., doubly periodic, periodic-Dirichlet) usually require \( f \) to be monotone in \( s \). The monotonicity enables one to work around difficulties created by the infinite multiplicity of the eigenvalue \( \lambda_0 = 0 \).

It is known that, with our boundary conditions, (1.2) has a solution for each \( h \in H \) if \( f \) is monotone in \( s \) and is asymptotically between (and bounded away from) two successive eigenvalues (Mawhin [M.1]), or if \( f \) is monotone in \( s \) and “jumps” (asymptotically, going from \(-\infty\) to \(+\infty\)) from one eigenvalue to the next, or to the one below (provided neither is \( \lambda_0 = 0 \)) (Willem [W]).

These results have been recently unified and generalized to include nonuniformities in the avoidance of \( \Sigma \) (Mawhin and Ward [M-W.1, M-W.2]). Here we show that \( f \) may jump across arbitrarily many eigenvalues of finite multiplicity and (1.3) may still be solvable for all \( h \in H \).

2. Statement of results. The following is our main result.

**Theorem 1.** Let \( f: J \times \mathbb{R} \to \mathbb{R} \) satisfy the Carathéodory conditions and (1.1). Let \( h \in L^2(J) \) and suppose:

- (c1) \( f(t,x,s) \) is monotone nondecreasing in \( s \) for each \( (t,x) \in J \).
- (c2) There is a number \( a > 0 \) and a function \( \beta \in H \) such that

\[
|f(t,x,s)| \leq f(t,x,s) + a|s| + \beta(t,x)
\]

for all \( (t,x,s) \in J \times \mathbb{R} \).
- (c3) There is a number \( \eta_0 > 0 \) such that \( 0 < \eta_0 \leq \lim_{|s| \to \infty} s^{-1}f(t,x,s) \), uniformly in \( (t,x) \in J \).

Then there is a number \( \alpha_0 > 0 \) such that a weak solution to the doubly \( 2\pi \)-periodic problem for (1.2) exists whenever \( \alpha < \alpha_0 \).

**Remark 1.** One may instead assume \( f(t,x,s) \) is nonincreasing in \( s \) and the existence of \( \eta_0 < 0 \) with \( 0 > \eta_0 \geq \lim_{|s| \to \infty} s^{-1}f(t,x,s) \). One may also replace \( f \) in (c2) by \(-f\).

As a corollary we have the following result on jumping nonlinearities. Consider the equation

\[
(2.1) \quad u_{tt} - u_{xx} + \alpha_- u^- - \alpha_+ u^+ - g(t,x,u) = h(t,x)
\]

where \( u^+ = \max(u,0) \), \( u^- = \max(-u,0) \), and \( u = u^+ - u^- \). Suppose \( h \in L^2(J) \) and \( g: J \times \mathbb{R} \to \mathbb{R} \) satisfies the Carathéodory conditions and (1.1).

**Corollary 1.** Let \( \alpha_- \) and \( \alpha_+ \) be positive numbers. Suppose \( f(t,x,s) := -\alpha_- s^- + \alpha_+ s^+ + g(t,x,s) \) is monotone nondecreasing in \( s \) and

\[
\lim_{|s| \to \infty} s^{-1}g(t,x,s) = 0,
\]

uniformly for \( (t,x) \in s \). Then there is a number \( \alpha_0 > 0 \) such that (2.1) has a weak doubly \( 2\pi \)-periodic solution provided \( 0 < \alpha_- < \alpha_0 \) (or \( 0 < \alpha_+ < \alpha_0 \)). (\( \alpha_0 \) does not depend on \( \alpha_+ , g, \) or \( h \)).

**Remark 2.** A similar corollary is true with \( \alpha_- \), \( \alpha_+ \) negative and \( f(t,x,s) \) nonincreasing in \( s \).

**Remark 3.** We note that if \( \alpha_- < \alpha_+ \) then we may have \( |\alpha_- , \alpha_+ | \cap \Sigma \neq \emptyset \). Indeed, the interval \( |\alpha_- , \alpha_+ | \) may contain any finite number of positive eigenvalues.
REMARK 4. The corollary may be viewed as a surjectivity result concerning operators of the form \( Lu + \alpha_- u^- - \alpha_+ u^+ \), where \( L \) is the D'Alembertian realized in \( L^2(J) \) with our boundary conditions.

3. **Abstract formulation and proofs.** Let \( H = L^2(J) \) with inner product \( (u, v) = \int_J uv \, dt \, dx \) and corresponding norm \( \| \cdot \| \). Each \( u \in H \) has a representation as a Fourier series of the form

\[
    u = \sum_{(m,n) \in \mathbb{Z} \times \mathbb{Z}} \alpha_{mn} u_{mn},
\]

where \( u_{mn} = e^{i(mt+nx)} \) for \( (m,n) \in \mathbb{Z} \times \mathbb{Z} \) and \( \alpha_{m,n} \in \mathbb{C} \) with \( \alpha_{m,n} = \overline{\alpha_{-m,-n}} \) to make the sum real.

Let

\[
    D(L) = \{ u \in H : \sum_{(m,n) \in \mathbb{Z}^2} (n^2 - m^2)^2 |\alpha_{mn}|^2 < \infty \}.
\]

Define \( L : D(L) \subseteq H \rightarrow H \) by

\[
    Lu = \sum_{(m,n) \in \mathbb{Z} \times \mathbb{Z}} (n^2 - m^2) \alpha_{mn} u_{mn} \quad \text{for } u \in D(L).
\]

Let \( F : H \rightarrow H \) be the substitution operator defined by \( f \). It is known that \( u \) is a weak solution of the doubly \( 2\pi \)-periodic problem for (1.2) if and only if \( u \in D(L) \) and

\[
    Lu - F(u) = h \tag{3.1}
\]

**Proof of Theorem 1.** Let \( \lambda_1 \) be the first positive eigenvalue of \( L \) (i.e., smallest positive number in \( \Sigma \) so that \( \lambda_1 = 1 \)). Choose \( \varepsilon_0 > 0 \) so that \( \varepsilon_0 < \min(\lambda_1, \eta_0) \). For \( \lambda \in ]0,1[ \) consider the family of equations

\[
    Lu - (1 - \lambda)\varepsilon_0 u - \lambda F(u) = h. \tag{3.2}
\]

The operator \( (L - \varepsilon_0 I)^{-1} \) is not compact. Nevertheless, it follows from a theorem in Willem's paper [Wi] (or see [M.2]) that it suffices to show that all possible solutions of (3.2) are bounded in \( H \) independently of \( \lambda \in ]0,1[ \).

If \( (u, \lambda) \) is a solution of (3.2) with \( 0 < \lambda < 1 \) then

\[
    Lu - (1 - \lambda)\varepsilon_0 u - \lambda f(t, x, u) = \lambda h(t, x), \tag{3.3}
\]

and by taking inner products with 1 we derive, since \( (Lu, 1) = 0 \),

\[
    \lambda \int_J f(t, x, u) \, dt \, dx = -(1 - \lambda)\varepsilon_0 \int_J u \, dt \, dx - \lambda \int_J h(t, x) \, dt \, dx. \tag{3.4}
\]

Taking absolute values in (3.3) and using (c2) we have a.e. on \( J \):

\[
    |Lu(t, x)| \leq (1 - \lambda)\varepsilon_0 |u(t, x)| + \lambda f(t, x, u(t, x)) \\
    + \alpha |u(t, x)| + \beta(t, x) + |h(t, x)|.
\]

Integrating over \( J \) and using (3.4) we obtain

\[
    |Lu|_{L^1} \leq (2\varepsilon_0 + \alpha) |u|_{L^1} + C_1 \tag{3.5}
\]

where \( C_1 \) is a constant.
For $u \in H$ let us write $u = u_0 + u_1$ with $u_0 \in \ker L$ and $u_1 \in \ker L^\perp = \text{Range } L$. It is known (cf., e.g., [L or C-H]) that there is a constant $\mu > 0$ such that for $u = u_0 + u_1 \in D(L)$,

$$|u_1|_{L^\infty} \leq \mu |Lu_1|_{L^1}.$$ 

Thus for any solution $u$ of (3.2) we have

$$|u_1|_{L^\infty} \leq \mu |Lu_1|_{L^1} \leq \mu (2\varepsilon_0 + \alpha) |u|_{L^1} + \mu C_1$$

and

(3.6) $$|u_1|_{L^\infty} \leq (2\varepsilon_0 + \alpha) C_2 |u| + C_3$$

for some constants $C_2$ and $C_3$. Of course, by (3.5) we also have

(3.7) $$|Lu_1|_{L^1} \leq (2\varepsilon_0 + \alpha) C_4 |u| + C_1.$$ 

Taking the inner product of the expression on each side of (3.2) with $u$ we derive

(3.8) $$(1 - \lambda) \varepsilon_0 |u|^2 + \lambda \int f(t,x,u)u = (Lu, u) - \lambda (h, u).$$ 

By condition (c3) there is a number $r > 0$ such that

$$f(t,x,s)s \geq \varepsilon_0 s^2$$

for $|s| \geq r$. Thus there is a function $\gamma \in H$ with

$$f(t,x,s)s \geq \varepsilon_0 s^2 - \gamma(t,x)$$

for all $(t, x, s) \in J \times \mathbb{R}$. From (3.8) we see that

$$(1 - \lambda) \varepsilon_0 |u|^2 + \lambda \int \varepsilon_0 |u|^2 dt dx - \int \gamma dt dx \leq |(Lu, u)| + |h| \cdot |u|.$$ 

Thus there is a constant $C$ with

$$\varepsilon_0 |u|^2 \leq |(Lu_1, u_1)| + |h| \cdot |u| + C \leq |Lu_1|_{L^1} \cdot |u_1|_{L^\infty} + |h| \cdot |u| + C.$$ 

By (3.6) and (3.7) we now obtain

$$\varepsilon_0 |u|^2 \leq [(2\varepsilon_0 + \alpha) C_4 |u| + C_1][(2\varepsilon_0 + \alpha) C_2 |u| + C_3] + |h| \cdot |u| + C$$

and, hence,

$$\varepsilon_0 |u|^2 \leq (2\varepsilon_0 + \alpha)^2 k_1 |u|^2 + k_2 |u| + k_3$$

for some constants $k_1, k_2, \text{and } k_3$. A subtraction yields

(3.9) $$[\varepsilon_0 - (2\varepsilon_0 + \alpha)^2 k_1] |u|^2 \leq k_2 |u| + k_3.$$ 

By now choosing $\varepsilon_0$ and $\alpha_0$ sufficiently small we can insure that, since $\alpha < \alpha_0$,

$$\varepsilon_0 - (2\varepsilon_0 + \alpha)^2 k_1 > 0,$$

which, by (3.9), implies $|u| < M$ for some constant $M > 0$. All possible solutions of (3.2) are thus bounded independently of $\lambda \in ]0,1[$, and (3.1) has a solution.

PROOF OF THE COROLLARY. We take

$$f(t,x,s) = -\alpha_- s^- + \alpha_+ s^+ + g(t, x, s).$$
By hypothesis $f$ is monotone nondecreasing in $s$. Also
$$|f(t, x, s)| \leq \alpha_- s^- + \alpha_+ s^+ + |g(t, x, s)|$$
$$\leq f(t, x, s) + 2\alpha_- s^- + 2|g(t, x, s)|.$$  

By hypothesis, for each $\varepsilon > 0$ there exists $\gamma_\varepsilon \in H$ with $|g(t, x, s)| \leq \varepsilon |s| + \gamma_\varepsilon(t, x)$. It follows that
$$|f(t, x, s)| \leq f(t, x, s) + (2\alpha_- + 2\varepsilon)|s| + 2\gamma_\varepsilon(t, x),$$
which shows (c2) holds. Since $\varepsilon > 0$ may be chosen arbitrarily small we can insure that $2\alpha_- + 2\varepsilon < \alpha_0$, where $\alpha_0$ is the number in Theorem 1, by requiring $\alpha_- < \alpha_0/2$ and then choosing $\varepsilon$. Since
$$\lim_{|s| \to \infty} s^{-1} f(t, x, s) \geq \min(\alpha_-, \alpha_+) > 0,$$
the corollary follows.

REMARK 5. Instead of looking for solutions $2\pi$-periodic in $t$ and $x$ we could also formulate our results for solutions $\omega_1$-periodic in $t$ and $\omega_2$-periodic in $x$ if we insist that $\omega_1/\omega_2$ be a rational number. This would insure that the d’Alembertian with these boundary conditions is realized in $H$ by a selfadjoint operator having properties like those of $L$ above. If $\omega_1/\omega_2$ is irrational, small divisors appear in the right inverse of $L$ which lead to unsolved difficulties.

4. A counterexample. It is easy to see that the corollary is false if $\alpha_- = 0$ and $\alpha_+ > 0$. For example, if $g \equiv 0$ we have

$$(4.1) \quad Lu - \alpha^+ u^+ = h,$$

and by taking inner products with $1$ we see that $h$ must satisfy $(h, 1) \leq 0$. Thus (4.1) cannot be solvable for all $h \in H$. In spite of this, (4.1) is certainly solvable for some $h \in H$. One might expect a solution if

$$(h, 1) = \int_J h \, dt \, dx < 0.$$

We show however that there may not be a solution even then.

It is easy to show that $u \in \ker L$ if and only if $u = p(t + x) + q(t - x)$ for some $p, q$ each $2\pi$-periodic on $\mathbb{R}$ with $p, q \in L^2(0, 2\pi)$.

Let $0 < \delta_1 < \delta_2 < \pi$ and $p: [0, 2\pi] \to \mathbb{R}$ be at least $C^2$ smooth and defined by

$$p(s) = \begin{cases} 
3 & \text{if } |\pi - s| \leq \delta_1, \\
0 & \text{if } |\pi - s| \geq \delta_2 \text{ and } 0 \leq s \leq 2\pi, \\
0 \leq p(s) \leq 3 & \text{elsewhere}.
\end{cases}$$

Extend $p$ $2\pi$-periodically to all of $\mathbb{R}$ and define $\phi \in H$ by

$$\phi(t, x) = p(t + x) \quad \text{for } (t, x) \in J.$$  

Then $\phi \in \ker L$; indeed, $\phi$ is a smooth (classical) solution of $u_{tt} - u_{xx} = 0$ and $\phi$ is $2\pi$-periodic in each of $x$ and $t$.

By choosing $\delta_2$ sufficiently small we can insure that

$$\int_J \phi \, dt \, dx < 4\pi^2.$$
On the other hand, by choosing $\delta_1$ sufficiently close to $\delta_2$, one can insure that
\[ \int_{\mathcal{J}} \phi \, dt \, dx < \int_{\mathcal{J}} \phi^2 \, dt \, dx. \]

Now consider (4.1) with $h = \phi - 1$. We observe that
\[ \int_{\mathcal{J}} h \, dt \, dx = \int_{\mathcal{J}} \phi \, dt \, dx - 4\pi^2 < 0. \]

Suppose $u \in D(L)$ solves
\[ (4.2) \quad Lu - \alpha_+ u^+ = \phi - 1. \]

Taking inner products with $\phi$ we obtain, since $(Lu, \phi) = 0$,
\[ 0 \geq -\alpha_+ \int_{\mathcal{J}} u^+ \phi \, dt \, dx = \int_{\mathcal{J}} (\phi - 1) \phi \, dt \, dx > 0. \]

Thus (4.2) cannot have a solution.

If we now let $h = \mu(\phi - 1)$ with $\mu \gg 1$ we see that there still is not a solution even when $-\int_{\mathcal{J}} h$ is large.

REFERENCES


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