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A CHARACTERIZATION OF THE WEYL SPECTRUM

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ABSTRACT. It is shown that for each closed subset Ω of the semi-Fredholm domain of a bounded linear operator T acting in a complex Hilbert space H there exists a subspace of a finite codimension in H such that the compression of $T - \lambda$ to this subspace is a left- or right-invertible operator for all λ in Ω. From this result we obtain a characterization of the Weyl spectrum of T.

Throughout this paper H denotes an infinite-dimensional Hilbert space with a scalar product $\langle \cdot, \cdot \rangle$, and $L(H)$ denotes the algebra of all bounded linear operators acting in H. For $T \in L(H)$, $\sigma(T)$ denotes the spectrum of T, $\rho(T)$ the resolvent set, $\rho_{\text{s-F}}(T)$ the semi-Fredholm domain of T, i.e. the set of those complex numbers λ such that $T - \lambda$ is a semi-Fredholm operator. We let

$$\operatorname{nul} T = \dim \ker T, \qquad \operatorname{min\,ind} T = \min\{\operatorname{nul} T, \operatorname{nul} T^*\}$$

and define the set of regular points of the semi-Fredholm domain by

$$\rho^{\text{r}}_{\text{s-F}}(T) = \{\lambda \in \rho_{\text{s-F}}(T) : \operatorname{min\,ind}(T - \mu) \text{ is continuous at } \lambda\},$$

and the set of singular points by

$$\rho^{\text{s}}_{\text{s-F}}(T) = \rho_{\text{s-F}}(T) \backslash \rho^{\text{r}}_{\text{s-F}}(T).$$

By [**1**, Proposition 2.6] these definitions are consistent with those ones given by C. Apostol in [**1**]. The reader is referred to [**2**, Chapter IV] for definitions and properties of semi-Fredholm operators.

The Weyl spectrum of T, $\sigma_{\text{W}}(T)$, is defined as the complement of those numbers $\lambda \in \rho_{\text{s-F}}(T)$ such that

$$\operatorname{ind}(T - \lambda) = \operatorname{nul}(T - \lambda) - \operatorname{nul}(T - \lambda)^* = 0.$$

The Browder spectrum may be defined by

$$\sigma_{\text{B}}(T) = \sigma(T) \backslash (\rho^{\text{s}}_{\text{s-F}}(T) \cap \overline{\rho(T)}).$$

Let $P(H)$ denote the set of all orthogonal projections in H with a finite-dimensional kernel. For any $P \in P(H)$ the compression of T to the range of P is defined by $T_P = PT\,|_{PH}$ and is an operator from $L(PH)$.

REMARK 1. If $P \in P(H)$ and T is a semi-Fredholm operator then T_P is semi-Fredholm and $\operatorname{ind} T_P = \operatorname{ind} T$.

PROOF. From the obvious equality $\operatorname{nul} T_P = \operatorname{nul} PTP - \operatorname{nul} P$, and similarly for T^*, we conclude that $\operatorname{ind} PTP = \operatorname{ind} T_P$, and since PTP is a finite-dimensional perturbation of T, the thesis results from the index theorem. □

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LEMMA 1. *Suppose* $0 \in \rho^{\mathrm{r}}_{\mathrm{s\text{-}F}}(T)$ *and* $\lambda_i \in \rho_{\mathrm{s\text{-}F}}(T)$ $(i = 1, 2, 3, \ldots)$. *Then there exists* $P \in P(H)$ *such that* $\min\operatorname{ind} T_P = 0$, *and*

$$\min\operatorname{ind}(T - \lambda_i)_P \le \min\operatorname{ind}(T - \lambda_i), \qquad i = 1, 2, 3, \ldots.$$

PROOF. Let $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ be orthonormal sets in $\ker T$ and $\ker T^*$, respectively. It follows from [**1**, Lemma 2.1] that $\ker T$ is orthogonal to $\ker T^*$. Putting

$$P_\alpha = \sum_{j=1}^{n} \langle \cdot, g_j^\alpha \rangle g_j^\alpha, \quad \text{where } g_j^\alpha = \sqrt{1-\alpha}\, e_j + \sqrt{\alpha} f_j,$$

we see that P_α is an orthogonal projection for any $\alpha \in [0, 1]$.

Now we fix $\lambda \in \rho_{\mathrm{s\text{-}F}}(T)$, $\lambda \neq 0$, and look for the conditions under which $\operatorname{nul}(T - \lambda)_P \le \operatorname{nul}(T - \lambda)$. Let N be the linear subspace spanned by $e_1, \ldots, e_n$ and put $M_\lambda = (N + \ker(T - \lambda))^\perp + N$. Since $M_\lambda \cap \ker(T - \lambda) = \{0\}$ and $M_\lambda + \ker(T - \lambda) = H$, we see that the operator $(T - \lambda) \mid_M \in L(M_\lambda, \operatorname{ran}(T - \lambda))$ is invertible and we put

$$T_\lambda = ((T - \lambda) \mid_M)^{-1} Q_\lambda - \lambda^{-1}(1 - Q_\lambda),$$

where Q_λ denotes the orthogonal projection on $\operatorname{ran}(T - \lambda)$. We have the equalities

$$T_\lambda e_j = -\lambda^{-1} e_j, \quad T_\lambda^* f_j = -\overline{\lambda}^{-1} f_j, \qquad j = 1, 2, \ldots, n. \tag{1}$$

The first follows from $e_j \in M_\lambda \cap \operatorname{ran}(T - \lambda)$, the second is implied by the identity

$$\begin{aligned} \langle y, f_j \rangle &= \langle (T - \lambda) T_\lambda Q_\lambda y \lambda T_\lambda (1 - Q_\lambda) y, f_j \rangle \\ &= \langle Q_\lambda y, T_\lambda^* (T - \lambda)^* f_j \rangle - \langle (1 - Q_\lambda) y, \overline{\lambda} T_\lambda^* f_j \rangle \\ &= \langle y, -\overline{\lambda} T_\lambda^* f_j \rangle. \end{aligned}$$

Suppose now that $(1 - P_\alpha)(T - \lambda)x = 0$. Then

$$(T - \lambda)x = P_\alpha (T - \lambda)x = \sum_{j=1}^{n} \beta_j g_j^\alpha,$$

for some complex numbers β_j. Writing x in the form $x = y - z$, where $y \in M_\lambda$, $z \in \ker(T - \lambda)$, we have $y = T_\lambda \sum_{j=1}^{n} \beta_j g_j^\alpha$, and the condition $x \in \operatorname{ran}(1 - P_\alpha)$ is equivalent to

$$\langle x, g_k^\alpha \rangle = \langle y - z, g_k^\alpha \rangle = 0 \qquad (k = 1, 2, \ldots, n)$$

or, equivalently,

$$\sum_{j=1}^{n} \beta_j \langle T_\lambda g_j^\alpha, g_k^\alpha \rangle = \langle z, g_k^\alpha \rangle, \qquad k = 1, 2, \ldots, n. \tag{2}$$

Because of (1) and the orthogonality of the vectors e_j and f_k, we have

$$\begin{aligned} \langle T_\lambda g_j^\alpha, g_k^\alpha \rangle =& (1 - \alpha) \langle T_\lambda e_j, e_k \rangle + \alpha \langle f_j, T_\lambda^* f_k \rangle \\ &+ \sqrt{\alpha(1 - \alpha)} (\langle T_\lambda f_j, e_k \rangle + \langle T_\lambda e_j, f_k \rangle) \\ =& -\lambda^{-1} \delta_{j,k} + \sqrt{\alpha(1 - \alpha)} \langle T_\lambda f_j, e_k \rangle. \end{aligned}$$

Let $\mathcal{T}_\lambda$ be the matrix $((\langle T_\lambda f_j, e_k\rangle))_{j,k=1}^n$. Then the conditions (2) may be written in the form

$$\left(\mathcal{T}_\lambda - \frac{I}{\lambda\sqrt{\alpha(1-\alpha)}}\right)\begin{pmatrix}\beta_1\\ \vdots\\ \beta_n\end{pmatrix} = \frac{1}{\sqrt{\alpha(1-\alpha)}}\begin{pmatrix}\langle z, g_1^\alpha\rangle\\ \vdots\\ \langle z, g_n^\alpha\rangle\end{pmatrix}. \tag{3}$$

When z varies over all $\ker(T-\lambda)$ the vectors on the right-hand side form a space of dimension at most $\mathrm{nul}(T-\lambda)$. Thus when the matrix $\mathcal{T}_\lambda - (\lambda\sqrt{\alpha(1-\alpha)})^{-1}I$ is invertible to each $z \in \ker(T-\lambda)$, there corresponds at most one vector $(\beta_1,\dots,\beta_n)$ satisfying (3) or, equivalently, an eigenvector $x = T_\lambda\sum_{j=1}^n \beta_j g_j^\alpha - z$ of $(T-\lambda)_{(1-P_\alpha)}$ which depends linearly on z. This then shows that $\mathrm{nul}(T-\lambda)_{(1-P_\alpha)} \le \mathrm{nul}(T-\lambda)$. Since for each λ the matrix $\mathcal{T}_\lambda$ has a finite number of eigenvalues, the set of those $\alpha \in (0,1)$ for which $\mathrm{nul}(T-\lambda_i)_{(1-P_\alpha)} > \mathrm{nul}(T-\lambda_i)$ (for at least one i) is at most countable. We may repeat our argument for the operator $(T-\lambda)^*$. We obtain

$$\min\mathrm{ind}(T-\lambda_i)_{(1-P_\alpha)} \le \min\mathrm{ind}(T-\lambda_i), \qquad i = 1,2, \tag{4}$$

for all $\alpha \in (0,1)$ with the possible exception of a countable set.

It suffices to prove the lemma in the case $\mathrm{nul}\,T \le \mathrm{nul}\,T^*$. Then $\mathrm{nul}\,T < \infty$ and we can assume $N = \ker T$. Suppose $(1-P_\alpha)Tx = 0$ and $x \in \mathrm{ran}(1-P_\alpha)$ for some $\alpha \in (0,1)$. Then $Tx = P_\alpha Tx = \sum_{j=1}^n \gamma_j g_j^\alpha$, for some complex numbers $\gamma_1,\dots,\gamma_n$. Since $\mathrm{ran}\,T$ is orthogonal to $\ker T^*$, we have $0 = \langle Tx, f_j\rangle = \sqrt{\alpha}\gamma_j$, $j = 1,2,\dots,n$. This implies $x \in \ker T \cap \mathrm{ran}(1-P_\alpha)$. Since $(1-P_\alpha)e_j = \alpha e_j - \sqrt{\alpha(1-\alpha)}f_j$ we have

$$x = \sum_{j=1}^n \langle x, e_j\rangle e_j = \sum_{j=1}^n \langle x, e_j\rangle(\alpha e_j - \sqrt{\alpha(1-\alpha)}f_j).$$

From the linear independence of vectors $e_1,\dots,e_n, f_1,\dots,f_n$ we get $x = 0$. Thus we have shown $\mathrm{nul}\,T_{(1-P_\alpha)} = 0$ for all $\alpha \in (0,1)$. Since we can find $\alpha \in (0,1)$ such that (4) holds, we see that $P = 1 - P_\alpha$ satisfies the thesis. □

LEMMA 2. *Let G_i $(1 \le i < n)$ be all the connected components of $\rho_{\text{s-F}}(T)$. Then there exist projections $P_i \in P(H)$ such that for all i $(1 \le i < n)$ we have:*

(i) $\mathrm{ran}\,P_{i+1} \subset \mathrm{ran}\,P_i$;

(ii) $\min\mathrm{ind}(T-\lambda)_{P_i} = 0$ for all $\lambda \in \bigcup_{j=1}^i G_j \cap \rho_{\text{s-F}}^{\mathrm{r}}(T_{P_i})$;

(iii) $\min\mathrm{ind}(T-\lambda)_{P_i} \le \min\mathrm{ind}(T-\lambda)_{P_{i-1}}$ for all $\lambda \in \rho_{\text{s-F}}^{\mathrm{r}}(T_{P_i})$;

(iv) $\sigma_{\mathrm{B}}(T_{P_i}) \subset \sigma_{\mathrm{B}}(T_{P_{i-1}})$;

(v) $\bigcap_{1\le j<n}\sigma_{\mathrm{B}}(T_{P_j}) = \sigma_{\mathrm{W}}(T)$,

where we have put $P_0 = I$.

PROOF. We may construct one by one the desired projections P_j $(1 \le j < n)$ as follows. Selecting numbers $\lambda_{i,j} \in G_i \cap \rho_{\text{s-F}}^{\mathrm{r}}(T_{P_{j-1}})$ we may find by Lemma 1 a projection $P_j \in P(H)$ such that $\mathrm{ran}\,P_j \subset \mathrm{ran}\,P_{j-1}$, $\min\mathrm{ind}(T-\lambda_{j,j})_{P_j} = 0$ and $\min\mathrm{ind}(T-\lambda_{i,j})_{P_j} \le \min\mathrm{ind}(T-\lambda_{i,j})_{P_{j-1}}$ $(1 \le i < n)$.

These relations and the facts listed below, which hold for any $P \in P(H)$, show that the thesis of the lemma is true.

(i) G_j $(1 \le j < n)$ are all the connected components of $\rho_{\text{s-F}}(T_P)$.

(ii) The functions $\mathrm{nul}(T-\lambda)_P$ and $\mathrm{nul}(T-\lambda)_P^*$ are constant on each of the sets $G_j \cap \rho_{\text{s-F}}^{\mathrm{r}}(T_P)$.

(iii) The function $\operatorname{ind}(T-\lambda)_P = \operatorname{ind}(T-\lambda)$ is constant on each G_j.

(iv) $\min\operatorname{ind}(T-\mu)_P > \min\operatorname{ind}(T-\lambda)_P$ for any $\mu \in \rho^{\mathrm{s}}_{\mathrm{s\text{-}F}}(T_p)\cap G_j$, $\lambda \in \rho^{\mathrm{r}}_{\mathrm{s\text{-}F}}(T_P)\cap G_j$.

(v) For each $\lambda \in \rho_{\mathrm{s\text{-}F}}(T)$ the equality $\operatorname{ind}(T-\lambda)_P = \min\operatorname{ind}(T-\lambda)_P = 0$ is equivalent to $\lambda \in \rho(T_P)$.

For the proof of these facts the reader is referred to [**2**, Chapter IV, §5] or [**1**]. □

THEOREM 1. *Suppose* $T \in L(H)$ *and* Ω *is a closed subset of* $\rho_{\mathrm{s\text{-}F}}(T)$. *Then there exists* $P \in P(H)$ *such that* $\min\operatorname{ind}(T-\lambda)_P = 0$ *for all* $\lambda \in \Omega$.

PROOF. Since there is only one unbounded component of $\rho_{\mathrm{s\text{-}F}}(T)$ we can find a finite number of connected components of $\rho_{\mathrm{s\text{-}F}}(T)$ which cover Ω. By Lemma 2 there exists $Q \in P(H)$ such that $\min\operatorname{ind}(T-\lambda)_Q = 0$ for all $\lambda \in \Omega \cap \rho^{\mathrm{r}}_{\mathrm{s\text{-}F}}(T_Q)$. The set $\Omega \cap \rho^{\mathrm{s}}_{\mathrm{s\text{-}F}}(T_Q)$ is finite and, by [**1**, Theorem 3.3], there exist two invariant subspaces Y, Z of T_Q such that $Y + Z = QH$, $Y \cap Z = \{0\}$, $\dim Z < \infty$ and

$$\rho^{\mathrm{r}}_{\mathrm{s\text{-}F}}(T_Q|_Y) = \rho^{\mathrm{r}}_{\mathrm{s\text{-}F}}(T_Q) \cup (\Omega \cap \rho^{\mathrm{s}}_{\mathrm{s\text{-}F}}(T_Q)).$$

Let P be the orthogonal projection on Y. Then $P \in P(H)$, $T_P = T_Q|_Y$ and, therefore, $\operatorname{nul}(T-\lambda)_P \le \operatorname{nul}(T-\lambda)_Q$ and $\operatorname{ind}(T-\lambda)_P = \operatorname{ind}(T-\lambda)_Q$ for all $\lambda \in \rho_{\mathrm{s\text{-}F}}(T_Q)$. Hence we have also

$$\min\operatorname{ind}(T-\lambda)_P \le \min\operatorname{ind}(T-\lambda)_Q = 0 \quad \text{for all } \lambda \in \rho^{\mathrm{r}}_{\mathrm{s\text{-}F}}(T_Q).$$

Now the thesis results from the continuity of the function $\min\operatorname{ind}(T-\lambda)|_P$ in the set Ω contained in $\rho^{\mathrm{r}}_{\mathrm{s\text{-}F}}(T_P)$. □

Theorem 2 is an obvious consequence of Theorem 1.

THEOREM 2. *For any* $T \in L(H)$ *we have*

$$\bigcap_{P\in P(H)} \sigma(T_P) = \sigma_{\mathrm{W}}(T). \quad \square$$

[**3**, Theorem 2.1] claims that the formula on the right-hand side in Theorem 2 characterizes the Browder spectrum of T. The discovery that [**3**, Theorem 2.1] is false was in inspiration for this paper. The mistake in [**3**] was already mentioned in [**4**].

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