

ASYMPTOTICS OF DIFFERENTIAL SYSTEMS WITH DEVIATING ARGUMENTS

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ABSTRACT. Conditions are given which imply that a system $X'(t) = F[t, X(g(t))]$ has a solution with some components which approach given limits with specified orders of convergence as $t \rightarrow \infty$, while the other components have specified orders of magnitude. The integral smallness conditions on F permit conditional convergence of some of the improper integrals that occur, and it is not required that $\lim_{t \rightarrow \infty} g(t) = \infty$.

1. Introduction. We consider the $n \times n$ system with deviating argument

$$(1) \quad x'_i(t) = f_i[t, X(g(t))], \quad 1 \leq i \leq n,$$

where $X = (x_1, \dots, x_n)$. The following assumption holds throughout.

ASSUMPTION A. ψ_1, \dots, ψ_n are positive and continuous on $[a, \infty)$, and g is continuous on $(0, \infty)$, with

$$(2) \quad \lim_{t \rightarrow \infty} g(t) = T_0 > a \geq 0.$$

The sets \mathcal{A} and \mathcal{B} form a partition of $\{1, \dots, n\}$, and (a) if $i \in \mathcal{A}$, then either $\psi_i = 1$ or $\psi_i(\tau)$ tends monotonically to zero as $\tau \rightarrow \infty$, (b) if $i \in \mathcal{B}$, then $\psi_i(\tau)$ tends monotonically to infinity as $\tau \rightarrow \infty$. All quantities are real, and $C = (c_1, \dots, c_n)$ is a given constant vector.

We say that $X = X(t)$ is a solution of (1) on $[t_0, \infty)$ if X is continuous for $t \geq \min\{t_0, \hat{t}_0\}$, where $\hat{t}_0 = \inf\{g(s) \mid s \geq t_0\}$, and satisfies (1) for $t > t_0$. We give conditions implying that if t_0 is sufficiently large, then (1) has a solution \bar{X} on $[t_0, \infty)$, such that

$$(3) \quad \begin{cases} \bar{x}_i(t) = c_i + O(\psi_i(t)), & i \in \mathcal{A}, \\ \bar{x}_i(t_0) = c_i \text{ and } \bar{x}_i(t) = O(\psi_i(t)), & i \in \mathcal{B}, \end{cases} \quad (t \rightarrow \infty),$$

where "O" may be replaced by "o" in some cases.

Our integral smallness conditions on f_1, \dots, f_n permit conditional convergence of some of the improper integrals that occur. This continues a theme developed in [5] with $g(t) = t$ and $\mathcal{A} = \{1, \dots, n\}$. (For earlier results on systems of ordinary differential equations subject to possibly conditional integral smallness conditions, see Hallam [1, 2].)

Although much has been written on the asymptotic properties of systems of ordinary differential equations, and on scalar equations with deviating arguments, relatively little has been published on systems with deviating arguments. (For examples, see Kitamura and Kusano [3, 4].)

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2. The main theorem. If h_1, \dots, h_n are locally integrable on $[\tau, \infty)$ and $T \geq \tau \geq a$, let

$$I(t; h_i) = \left| \int_t^\infty h_i(s) ds \right|, \quad i \in \mathcal{A};$$

$$I(t; h_i) = \left| \int_T^t h_i(s) ds \right|, \quad i \in \mathcal{B}.$$

We will state assumptions below of the form

$$(4) \quad \overline{\lim}_{t \rightarrow \infty} (\psi_i(t))^{-1} I(t; h_i) = \alpha_i < \infty, \quad 1 \leq i \leq n.$$

In all cases it is implicitly assumed that $\int^\infty h_i(s) ds$ converges for $i \in \mathcal{A}$, perhaps conditionally if h_i is not eventually sign constant. Obviously, $\alpha_i = 0$ if $i \in \mathcal{A}$ and $\psi_i = 1$.

REMARK 1. The constants $\alpha_1, \dots, \alpha_n$ in (4) are independent of T . This is obvious if $i \in \mathcal{A}$; if $i \in \mathcal{B}$, it follows from the assumption that $\lim_{t \rightarrow \infty} \psi_i(t) = \infty$. If $\varepsilon > 0$ and $i \in \mathcal{B}$, then (4) implies that there is a $t_0 \geq T$ such that

$$\left| \int_{t_0}^t h_i(s) ds \right| \leq 2(\alpha_i + \varepsilon)\psi_i(t), \quad t \geq t_0,$$

and the two may be dropped if h_i is sign constant on $[T, \infty)$.

THEOREM 1. Suppose there are constants T, M_1, \dots, M_n such that

$$(5) \quad a < T < T_0; \quad M_i > 0, \quad 1 \leq i \leq n,$$

$g(t) \geq a$ ($t \geq T$), and f_1, \dots, f_n are continuous and satisfy

$$(6) \quad |f_i(t, Y) - f_i(t, C)| \leq w_i(t, |y_1 - c_1|, \dots, |y_n - c_n|), \quad 1 \leq i \leq n,$$

on the set

$$(7) \quad \Omega = \{(t, Y) \mid t \geq T; |y_j - c_j| \leq M_j \psi_j(g(t)), 1 \leq j \leq n\},$$

where $w_i = w_i(t, \xi_1, \dots, \xi_n)$ is continuous for $t \geq T$, $\xi_j \geq 0$ ($1 \leq j \leq n$), and nondecreasing in ξ_1, \dots, ξ_n . Define

$$(8) \quad \sigma_i(t) = w_i(t, M_1 \psi_1(g(t)), \dots, M_n \psi_n(g(t))), \quad t \geq T.$$

Suppose also that

$$(9) \quad \overline{\lim}_{t \rightarrow \infty} (\psi_i(t))^{-1} I(t; f_i(\cdot; C)) = A_i < \infty, \quad 1 \leq i \leq n,$$

and

$$(10) \quad \overline{\lim}_{t \rightarrow \infty} (M_i \psi_i(t))^{-1} I(t; \sigma_i) = \theta_i < 1, \quad 1 \leq i \leq n,$$

where

$$(11) \quad M_i > A_i/(1 - \theta_i), \quad i \in \mathcal{A}; \quad M_i > 2A_i/(1 - \theta_i), \quad i \in \mathcal{B}.$$

Then, if t_0 is sufficiently large, (1) has a solution \bar{X} on $[t_0, \infty)$ such that

$$(12) \quad |\bar{x}_i(t) - c_i| \leq M_i \psi_i(t), \quad t \geq t_0, \quad 1 \leq i \leq n; \quad \bar{x}_i(t_0) = c_i, \quad i \in \mathcal{B};$$

and

$$(13) \quad \overline{\lim}_{t \rightarrow \infty} (\psi_i(t))^{-1} |\bar{x}_i(t) - c_i| \leq A_i + M\theta_i, \quad 1 \leq i \leq n.$$

PROOF. Choose $t_0 \geq T$ so that

$$(14) \quad g(t) \geq T, \quad t \geq t_0$$

(see (2) and (5)), and so that the functions

$$(15) \quad \mu_i(t) = \begin{cases} \int_{t_0}^{\infty} \sigma_i(s) ds + \left| \int_{t_0}^{\infty} f_i(s, C) ds \right|, & i \in \mathcal{A}, \\ \int_{t_0}^t \sigma_i(s) ds + \left| \int_{t_0}^t f_i(s, C) ds \right|, & i \in \mathcal{B}, \end{cases}$$

satisfy the inequalities

$$(16) \quad \mu_i(t) \leq M_i \psi_i(t), \quad t \geq t_0.$$

This is possible because of (9)–(11). (See Remark 1.) Let \mathcal{C}_T be the space of continuous n -vector functions on $[T, \infty)$ with the topology of uniform convergence on finite intervals, and let

$$(17) \quad \mathcal{M} = \{X \in \mathcal{C}_T \mid |x_i(\tau) - c_i| \leq M_i \psi_i(\tau), \tau \geq T, 1 \leq i \leq n\}.$$

Define the transformation $\hat{X} = \mathcal{T}X$ by

$$(18) \quad \hat{x}_i(t) = \begin{cases} c_i - \int_{t_0}^{\infty} f_i[s, X(g(s))] ds, & t \geq t_0, \\ c_i - \int_{t_0}^{\infty} f_i[s, X(g(s))] ds, & T \leq t \leq t_0, \end{cases} \quad i \in \mathcal{A};$$

$$(19) \quad \hat{x}_i(t) = \begin{cases} c_i + \int_{t_0}^t f_i[s, X(g(s))] ds, & t \geq t_0, \\ c_i, & T \leq t \leq t_0, \end{cases} \quad i \in \mathcal{B}.$$

We will use the Schauder-Tychonoff theorem to show that $\bar{X} = \mathcal{T}\bar{X}$ for some \bar{X} in \mathcal{M} , and then verify that \bar{X} has the stated properties.

Clearly \mathcal{M} is a closed convex subset of the Fréchet space \mathcal{C}_T . If $X \in \mathcal{M}$, then (14) and (17) imply that $(t, X(g(t))) \in \Omega$ for $t \geq t_0$, so the functions $h_i(t) = f_i[t, X(g(t))]$ are continuous on $[t_0, \infty)$, and (6), (7), and the monotonicity of w_i imply that

$$(20) \quad |f_i[t, X(g(t))] - f_i(t, C)| \leq \sigma_i(t), \quad t \geq t_0, 1 \leq i \leq n, X \in \mathcal{M}.$$

By writing

$$\int f_i[s, X(g(s))] ds = \int (f_i[s, X(g(s))] - f_i(s, C)) ds + \int f_i(s, C) ds,$$

it is easily seen from (15), (18), (19), and (20) that

$$(21) \quad |\hat{x}_i(t) - c_i| \leq \mu_i(t) \leq M_i \psi_i(t), \quad t \geq t_0, 1 \leq i \leq n,$$

where the second inequality comes from (16). Since $\psi_i(t) \geq \psi_i(t_0)$ if $T \leq t \leq t_0$ and $i \in \mathcal{A}$, (21) and the definition of \hat{X} on this interval imply that $\hat{X} \in \mathcal{M}$; thus, \mathcal{T} maps \mathcal{M} into itself.

If X and X_j are in \mathcal{M} , then

$$(22) \quad |\hat{x}_{ij}(t) - \hat{x}_i(t)| \leq \int_{t_0}^{\infty} |f_i[s, X_j(g(s))] - f_i[s, X(g(s))]| ds, \quad t \geq T,$$

if $i \in \mathcal{A}$ (see (18)), and

$$(23) \quad |\hat{x}_{ij}(t) - \hat{x}_i(t)| \leq \int_{t_0}^{t_1} |f_i[s, X_j(g(s))] - f_i[s, X(g(s))]| ds, \quad T \leq t \leq t_1,$$

if $i \in \mathcal{B}$ (see (19)). From (20), $2\sigma_i(s)$ majorizes the integrands in (22) and (23); hence, (10) and Lebesgue's dominated convergence theorem imply that if $\lim_{j \rightarrow \infty} X_j(t) = X(t)$ ($t \geq T$), then $\lim_{j \rightarrow \infty} \hat{x}_{ij}(t) = \hat{x}_i(t)$ ($t \geq T$) with uniform convergence on $[T, \infty)$ if $i \in \mathcal{A}$ and on $[T, t_1]$ for every $t_1 > T$ if $i \in \mathcal{B}$. Therefore, \mathcal{T} is continuous on \mathcal{H} .

If $X \in \mathcal{H}$, then (18) and (19) imply that $x'_i = 0$ on $[T, t_0]$, and

$$|x'_i(t)| = |f_i[t, X(g(t))]| \leq \sigma_i(t) + |f_i(t, C)|, \quad t \geq t_0, \quad 1 \leq i \leq n$$

(see (20)). Hence, the family $\mathcal{T}(\mathcal{H})$ is equicontinuous on compact subsets of $[T, \infty)$. Since $\mathcal{T}(\mathcal{H}) \subset \mathcal{H}$, (17) implies that $\mathcal{T}(\mathcal{H})$ is uniformly bounded on such subsets. Therefore, the Arzela-Ascoli theorem implies that $\mathcal{T}(\mathcal{H})$ has compact closure.

The Schauder-Tychonov theorem now implies that $\overline{\mathcal{X}} = \mathcal{T}\overline{\mathcal{X}}$ for some $\overline{\mathcal{X}}$ in \mathcal{H} . Setting $x_i = \hat{x}_i = \bar{x}_i$ in (18) and (19) shows that $\overline{\mathcal{X}}$ is a solution of (1) on $[t_0, \infty)$ and $\bar{x}_i(t_0) = c_i$ if $i \in \mathcal{B}$. For (13), we observe that $|\bar{x}_i(t) - c_i| \leq \mu_i(t)$, because of (21) with $\hat{x}_i = \bar{x}_i$, and that $\overline{\lim}_{t \rightarrow \infty} (\psi_i(t))^{-1} \mu_i(t) \leq A_i + M\theta_i$, because of (9), (10), and (15). This completes the proof.

3. An application to a linear system. The linear system

$$(24) \quad x'_i(t) = \sum_{j=1}^n a_{ij}(t)x_j(g(t)) + h_i(t), \quad 1 \leq i \leq n,$$

is of the form (1), with

$$(25) \quad f_i(t, Y) = \sum_{j=1}^n a_{ij}(t)y_j + h_i(t).$$

Let

$$(26) \quad \lambda_i(t) = \sum_{j=1}^n |a_{ij}(t)|\psi_j(g(t)).$$

THEOREM 2. *Suppose that $a_{ij}, h_i \in C(0, \infty)$ and (9) holds, with f_i as in (25). Suppose also that*

$$(27) \quad \overline{\lim}_{t \rightarrow \infty} (\psi_i(t))^{-1} I(t; \lambda_i) = \theta_i < 1, \quad 1 \leq i \leq n.$$

Then (24) has a solution $\overline{\mathcal{X}}$ on $[t_0, \infty)$ which satisfies (3), provided t_0 is sufficiently large; moreover, if $A_i = \theta_i = 0$ for some i , then "O" may be replaced by "o" in (3) for that i .

PROOF. In this case (6) holds on $(0, \infty) \times R^n$, with

$$w_i(t, \xi_1, \dots, \xi_n) = \sum_{j=1}^n |a_{ij}(t)|\xi_j.$$

Let T be in (a, T_0) and $M_1 = \dots = M_n = M$, where M is chosen to satisfy (11); then (8) becomes $\sigma_i(t) = M\lambda_i(t)$ (see (26)), so (27) implies (10). Now Theorem 1 implies the conclusion.

EXAMPLE. Consider the system

$$(28) \quad \begin{bmatrix} x'_1(t) \\ x'_2(t) \end{bmatrix} = \begin{bmatrix} 2t^{-2} \sin t & t^{-1} \\ t^{-3} & t^{-3} \sin t \end{bmatrix} \begin{bmatrix} x_1(t^\nu) \\ x_2(t^\nu) \end{bmatrix} + \begin{bmatrix} 6t^{-1} \\ -2t^{-3} \end{bmatrix},$$

where $\nu > 0$. Here λ_1 and λ_2 in (26) are

$$(29) \quad \lambda_1(t) = 2t^{-2}|\sin t|\psi_1(t^\nu) + t^{-1}\psi_2(t^\nu),$$

$$(30) \quad \lambda_2(t) = t^{-3}\psi_1(t^\nu) + t^{-3}|\sin t|\psi_2(t^\nu).$$

If $C = (c_1, c_2)$, then

$$(31) \quad \int_T^t f_1(s, C) ds = (c_2 + 6) \log t + O(1)$$

and

$$(32) \quad \int_t^\infty f_1(s, C) ds = O(t^{-2}) \quad \text{if } c_2 = -6.$$

In any case,

$$(33) \quad \int_t^\infty f_2(s, C) ds = \frac{(c_1 - 2)t^{-2}}{2} + O(t^{-3}).$$

We first apply Theorem 2 with $\mathcal{A} = \{2\}$ and $\mathcal{B} = \{1\}$. If $\psi_1(t) = \log t$, then (31) implies (9) for $i = 1$ and (30) implies that

$$\int_t^\infty \lambda_2(s) ds = \frac{\nu}{2}t^{-2} \log t + O(t^{-2})$$

if ψ_2 is any nonincreasing function. Therefore, (27) holds for $i = 2$ if $\nu < 2$ and $\psi_2(t) = t^{-2} \log t$, or if $\nu \geq 2$ and $\psi_2(t) = t^{-2+\alpha} \log t$, where α is any positive number; moreover, $\theta_2 = 0$ in the second case. Since (27) holds for $i = 1$ with $\theta_1 = 0$ in either case (if we assume, without loss of generality, that $\alpha < 2$), Theorem 2 implies that if t_0 is sufficiently large, then (28) has a solution \bar{X} on $[t_0, \infty)$ such that $\bar{x}_1(t_0) = c_1$, $\bar{x}_1(t) = O(\log t)$, and $\bar{x}_2(t) = c_2 + O(t^{-2} \log t)$ if $\nu < 2$, or $\bar{x}_2(t) = c_2 + o(t^{-2+\alpha} \log t)$ if $\nu \geq 2$.

If $c_2 = -6$, we can apply Theorem 2 with $\alpha = \{1, 2\}$ and \mathcal{B} empty. From (32) and (33), it seems reasonable to let $\psi_1(t) = t^{-a}$ and $\psi_2(t) = t^{-b}$, with a and b as large as possible. If $c_1 \neq 2$, then (32) and (33) dictate that $a \leq 2$ and $b \leq 2$. If λ_1 and λ_2 in (29) and (30) are to satisfy (27), then we must have $\nu a + 1 \geq a$, $\nu b \geq a$, $\nu a + 2 \geq b$, and $\nu b + 2 \geq b$. If $\nu > 1$, let $a = b = 2$; then (27) holds with $\theta_1 = \theta_2 = 0$, and so (28) has a solution \bar{X} such that $\bar{x}_1(t) = c_1 + O(t^{-2})$ and $\bar{x}_2(t) = -6 + O(t^{-2})$. If $\nu \leq 1$, let $a = 2\nu$ and $b = 2$; then (27) holds with $\theta_1 = \frac{1}{2}\nu$ and $\theta_2 = 0$, so (28) has a solution \bar{X} such that $\bar{x}_1(t) = c_1 + O(t^{-2\nu})$ and $\bar{x}_2(t) = -6 + O(t^{-2})$ if $1/2 < \nu \leq 1$. If $0 < \nu \leq 1/2$ and $\alpha > 0$, then (28) has a solution \bar{X} such that $\bar{x}_1(t) = c_1 + o(t^{-2\nu+\alpha})$ and $\bar{x}_2(t) = -6 + O(t^{-2})$. Sharper estimates can be obtained for the case where $C = (2, -6)$, but we omit the details.

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