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COMPLETELY BOUNDED HOMOMORPHISMS OF OPERATOR ALGEBRAS

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ABSTRACT. Let A be a unital operator algebra. We prove that if ρ is a completely bounded, unital homomorphism of A into the algebra of bounded operators on a Hilbert space, then there exists a similarity S, with $\|S^{-1}\| \cdot \|S\| = \|\rho\|_{\text{cb}}$, such that $S^{-1}\rho(\cdot)S$ is a completely contractive homomorphism. We also show how Rota's theorem on operators similar to contractions and the result of Sz.-Nagy and Foias on the similarity of ρ-dilations to contractions can be deduced from this result.

1. Introduction. In [**6**] we proved that a homomorphism ρ of an operator algebra is similar to a completely contractive homomorphism if and only if ρ is completely bounded. It was known that if S is such a similarity, then $\|S\| \cdot \|S^{-1}\| \geq \|\rho\|_{\text{cb}}$. However, at the time we were unable to determine if one could choose the similarity such that $\|S\| \cdot \|S^{-1}\| = \|\rho\|_{\text{cb}}$. When the operator algebra is a C^*-algebra then Haagerup had shown [**3**] that such a similarity could be chosen. The purpose of the present note is to prove that for a general operator algebra, there exists a similarity S such that $\|S\| \cdot \|S^{-1}\| = \|\rho\|_{\text{cb}}$.

Completely contractive homomorphisms are central to the study of the representation theory of operator algebras, since they are precisely the homomorphisms that can be dilated to a $*$-representation on some larger Hilbert space of any C^*-algebra which contains the operator algebra. For C^*-algebras the sets of contractive homomorphisms, completely contractive homomorphisms, and $*$-homomorphisms coincide.

The main result of this paper also gives, at least, a theoretical answer to certain minimization problems. Suppose, for example, that T is an operator on a Hilbert space that is similar to a contraction; then $\inf\{\|S\| \cdot \|S^{-1}\| : \|S^{-1}TS\| \leq 1\}$ is attained and equal to $\|\rho\|_{\text{cb}}$, where ρ is the homomorphism of the disk algebra defined by $\rho(f) = f(T)$. An extensive study of this infimum was undertaken in [**5**], and it was studied in [**2**] for certain Toeplitz operators.

Finally, we end this note by showing how Rota's theorem [**7**] that every operator with spectral radius less than 1 is similar to a contraction, and the result of Sz.-Nagy and Foias [**8**] that every operator with a ρ-dilation is similar to a contraction, can be easily deduced from our result. These new proofs give a unified principle of estimating the above infimum for both of these classes of operators.

2. The similarity theorem. Let H denote a Hilbert space, $L(H)$ the bounded linear operators on H, B a unital C^*-algebra, and let A be a subalgebra of B containing the unit of B. We call A an operator algebra. We let M_n denote the

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$n \times n$ complex matrices, and $M_n(A)$ the tensor product of A and M_n. We endow $M_n(A)$ with the norm that it inherits as a subspace of the C^*-algebra $M_n(B)$.

Given a map ρ: $A \to L(H)$, we define maps ρ_n: $M_n(A) \to L(H + \cdots + H)$ (n copies) by $\rho_n((a_{ij})) = (\rho(a_{ij}))$ for (a_{ij}) in $M_n(A)$. We call ρ *completely bounded* provided that $\sup_n \|\rho_n\|$ is finite and we let $\|\rho\|_{\text{cb}}$ denote this supremum. If $\|\rho\|_{\text{cb}} \leq 1$, then we say that ρ is *completely contractive*. By [**1**], a homomorphism ρ of an operator algebra A into $L(H)$ is completely contractive if and only if it can be *dilated* to B, that is, if and only if there exists a $*$-representation Π: $B \to L(K)$ for some Hilbert space K, containing H, such that $\rho(a) = P\Pi(a)|_H$ for all a in A, where P denotes the projection of K onto H.

THEOREM. *Let A be an operator algebra contained in a C^*-algebra B, and let ρ: $A \to L(H)$ be a unital, completely bounded homomorphism. Then there exists an invertible operator S, with $\|S\| \cdot \|S^{-1}\| = \|\rho\|_{\text{cb}}$ such that $S^{-1}\rho(\cdot)S$ is a completely contractive homomorphism.*

PROOF. By the generalization of Stinespring's Theorem [**6**, Theorem 2.8] and by [**6**, Theorem 2.4], there exists a Hilbert space K, a $*$-homomorphism Π: $B \to L(K)$, and two bounded operators V_i: $H \to K$, $i = 1, 2$, with $\|V_1\| \cdot \|V_2\| = \|\rho\|_{\text{cb}}$ such that $\rho(a) = V_1^*\Pi(a)V_2$, for a in A.

Following [**4**, p. 1030], for $h \in H$, we define

$$|h| = \inf\left\{\left\|\sum \Pi(a_i)V_2h_i\right\| : \sum \rho(a_i)h_i = h,\ a_i \in A,\ h_i \in H\right\},$$

where the infimum is taken over finite sums. By a minor modification of the arguments in [**4**, p. 1030], one obtains that $|\cdot|$ is a norm on H and $(H, |\cdot|)$ is a Hilbert space.

If $h = \sum \rho(a_i)h_i$, then

$$\begin{aligned} \|h\| &= \left\|\sum \rho(a_i)h_i\right\| = \left\|\sum V_1^*\Pi(a_i)V_2h_i\right\| \\ &\leq \|V_1^*\| \cdot \left\|\sum \Pi(a_i)V_2h_i\right\|. \end{aligned}$$

Thus, $\|h\| \leq \|V_1^*\| \cdot |h|$. Similarly, $\rho(1)h = h$ yields $|h| \leq \|V_2\| \cdot \|h\|$. Thus, if we define S: $(H, |\cdot|) \to (H, \|\cdot\|)$ to be the identity, then S is invertible and

$$\|S^{-1}\| \cdot \|S\| \leq \|V_1^*\| \cdot \|V_2\| = \|\rho\|_{\text{cb}}.$$

To complete the proof of the theorem it will be sufficient to prove that $S^{-1}\rho(\cdot)S$ is completely contractive, since then $\|S^{-1}\| \cdot \|S\| \geq \|\rho\|_{\text{cb}}$ necessarily.

Let $a \in A$, and let $h = \sum \rho(a_i)h_i$. Then

$$|\rho(a)h| \leq \left\|\sum \Pi(aa_i)V_2h_i\right\| \leq \|a\| \cdot \left\|\sum \Pi(a_i)V_2h_i\right\|,$$

so $|\rho(a)h| \leq \|a\| \cdot |h|$. Thus, we obtain that $S^{-1}\rho(\cdot)S$ is contractive.

To see that $S^{-1}\rho(\cdot)S$ is completely contractive, fix an integer n, let $\hat{H} = H + \cdots + H$ (n copies) and let $|\cdot|_n$ denote the norm on $\hat{H}$ given by $|\hat{h}|_n = |h_1|^2 + \cdots + |h_n|^2$, $\hat{h} = (h_1, \ldots, h_n)$. We must prove that if $\hat{a} = (a_{i,j}) \in M_n(A)$ then $|\rho_n(\hat{a})\hat{h}|_n \leq \|\hat{a}\| \cdot |\hat{h}|_n$ for $\hat{h} \in \hat{H}$.

To this end, consider ρ_n: $M_n(A) \to L(\hat{H})$ where $\hat{H}$ is endowed with its old norm, i.e., $\|\hat{h}\|^2 = \|h_1\|^2 + \cdots + \|h_n\|^2$. Since ρ is completely bounded, ρ_n will be completely bounded and, in fact, $\|\rho_n\|_{\rm cb} = \|\rho\|_{\rm cb}$. Thus, by the first part of our argument we may endow $\hat{H}$ with yet another norm $|\cdot|_{(n)}$ such that $\rho_n(\cdot)$ is contractive in this norm, i.e., $|\rho_n(\hat{a})\hat{h}|_{(n)} \leq \|\hat{a}\| \cdot |\hat{h}|_{(n)}$.

To construct $|\cdot|_{(n)}$, all we need is a Stinespring representation of ρ_n. For such a representation, consider Π_n: $M_n(B) \to L(\hat{K})$, $\hat{K} = K + \cdots + K$ (n copies) and $\hat{V}_i$: $\hat{H} \to \hat{K}$ defined by $\hat{V}_i(h_1, \ldots, h_n) = (V_i h_1, \ldots, V_i h_n)$, $i = 1, 2$. It is easily seen that $\rho_n(\hat{a}) = \hat{V}_1 \Pi_n(\hat{a}) \hat{V}_2$ for $\hat{a} \in M_n(A)$. Thus, we may set

$$|\hat{h}|_{(n)} = \inf\left\{ \left\| \sum \Pi_n(\hat{a}_i)\hat{V}_2\hat{h}_i \right\| : \sum \rho_n(\hat{a}_i)\hat{h}_i = \hat{h} \right\},$$

and ρ_n will be contractive in this norm.

We claim that with these choices $|\hat{h}|_{(n)} = |\hat{h}|_n$, which will complete the proof of the theorem.

To prove the claim fix $\varepsilon > 0$, let $\hat{a}_k = (a_{i,j,k}) \in M_n(A)$, $\hat{h}_k = (h_{1,k}, \ldots, h_{n,k}) \in \hat{H}$ be such that, $\sum \rho_n(\hat{a}_k)\hat{h}_k = \hat{h}$, and $|\hat{h}|^2_{(n)} + \varepsilon \geq \|\sum \Pi_n(\hat{a}_k)\hat{V}_2\hat{h}_k\|^2$. We then have that

$$|\hat{h}|^2_{(n)} + \varepsilon \geq \sum_{i=1}^{n} \left\| \sum_k \sum_{j=1}^{n} \Pi(a_{i,j,k}) V_2 h_{j,k} \right\|^2 \geq \sum_{i=1}^{n} |h_i|^2 = |\hat{h}|^2_n,$$

and so $|\hat{h}|_{(n)} \geq |\hat{h}|_n$. The other inequality follows similarly. This completes the proof of the theorem.

To see how Rota's Theorem [**7**] follows from the above, let T be an operator whose spectrum is contained in the open unit disk. Recall that by the Riesz functional calculus, if $f(z)$ is a polynomial, then

$$f(T) = \frac{1}{2\Pi i} \int_\Gamma f(z)(T - zI)^{-1}\, dz,$$

where $\Gamma = \{z\colon\ |z| = 1\}$. Setting $\rho(f) = f(T)$, and letting $\|f\| = \sup\{|f(z)|\colon\ |z| = 1\}$, we have that $\|\rho(f)\| \leq K\|f\|$,where

$$K = \frac{1}{2\Pi} \int_\Gamma \|(T - zI)^{-1}\|\, d|z|.$$

Thus, ρ extends to a bounded homomorphism of the disk algebra. To see that ρ is completely bounded (here we are thinking of the disk algebra as a subalgebra of the C^*-algebra of continuous functions on the circle), observe that for an $n \times n$ matrix of polynomials,

$$\begin{aligned}(f_{i,j}(T)) &= \frac{1}{2\Pi i} \int (f_{i,j}(z)(T - zI^{-1}))\, dz \\ &= \frac{1}{2\Pi i} \int (f_{i,j}(z))(\hat{T} - z\hat{I})\, dz,\end{aligned}$$

where $\hat{T}$ is the direct sum of n copies of T. Since $\|(T - zI)^{-1}\| = \|(\hat{T} - z\hat{I})^{-1}\|$, we have

$$\|(f_{i,j}(T))\| \leq K\|(f_{i,j}(z))\|,$$

and so ρ is completely bounded with $\|\rho\|_{\rm cb} \leq K$. Hence, there is an invertible operator S such that $\|S^{-1}\| \cdot \|S\| \leq K$ and $\|S^{-1}TS\| = \|S^{-1}\rho(z)S\| \leq \|z\| = 1$.

As a second application we mention the ρ-dilations considered in Sz.-Nagy and Foias [**8**]. An operator T in $L(H)$ has a ρ-dilation if there is a unitary U acting on K, H contained in K, such that $T^n = \rho PU^n|_H$, $n \geq 1$, where P is the projection of K onto H. For f in the disk algebra, define $\phi(f) = Pf(U)|_H$, and $\Psi(f) = f(0) \cdot I$. One easily sees that ϕ and Ψ are complete contractions.

Finally, setting $\gamma(f) = f(T) = \rho\phi(f) + (1-\rho)\Psi(f)$, we have that γ is a completely bounded homomorphism, and $\|\gamma\|_{\text{cb}} \leq 2\rho - 1$. Thus, there is an invertible S, $\|S^{-1}\| \cdot \|S\| \leq 2\rho - 1$, such that $S^{-1}TS$ is a contraction.

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