COMPLETELY BOUNDED HOMOMORPHISMS
OF OPERATOR ALGEBRAS

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ABSTRACT. Let \( A \) be a unital operator algebra. We prove that if \( \rho \) is a completely bounded, unital homomorphism of \( A \) into the algebra of bounded operators on a Hilbert space, then there exists a similarity \( S \), with \( \|S^{-1}\| \cdot \|S\| = \|\rho\|_{cb} \), such that \( S^{-1}\rho(S)S \) is a completely contractive homomorphism. We also show how Rota's theorem on operators similar to contractions and the result of Sz.-Nagy and Foias on the similarity of \( \rho \)-dilations to contractions can be deduced from this result.

1. Introduction. In [6] we proved that a homomorphism \( \rho \) of an operator algebra is similar to a completely contractive homomorphism if and only if \( \rho \) is completely bounded. It was known that if \( S \) is such a similarity, then \( \|S\| \cdot \|S^{-1}\| \geq \|\rho\|_{cb} \). However, at the time we were unable to determine if one could choose the similarity such that \( \|S\| \cdot \|S^{-1}\| = \|\rho\|_{cb} \). When the operator algebra is a \( C^* \)-algebra then Haagerup had shown [3] that such a similarity could be chosen. The purpose of the present note is to prove that for a general operator algebra, there exists a similarity \( S \) such that \( \|S\| \cdot \|S^{-1}\| = \|\rho\|_{cb} \).

Completely contractive homomorphisms are central to the study of the representation theory of operator algebras, since they are precisely the homomorphisms that can be dilated to a \( * \)-representation on some larger Hilbert space of any \( C^* \)-algebra which contains the operator algebra. For \( C^* \)-algebras the sets of contractive homomorphisms, completely contractive homomorphisms, and \( * \)-homomorphisms coincide.

The main result of this paper also gives, at least, a theoretical answer to certain minimization problems. Suppose, for example, that \( T \) is an operator on a Hilbert space that is similar to a contraction; then \( \inf\{\|S\| \cdot \|S^{-1}\| : \|S^{-1}TS\| \leq 1\} \) is attained and equal to \( \|\rho\|_{cb} \), where \( \rho \) is the homomorphism of the disk algebra defined by \( \rho(f) = f(T) \). An extensive study of this infimum was undertaken in [5], and it was studied in [2] for certain Toeplitz operators.

Finally, we end this note by showing how Rota's theorem [7] that every operator with spectral radius less than 1 is similar to a contraction, and the result of Sz.-Nagy and Foias [8] that every operator with a \( \rho \)-dilation is similar to a contraction, can be easily deduced from our result. These new proofs give a unified principle of estimating the above infimum for both of these classes of operators.

2. The similarity theorem. Let \( H \) denote a Hilbert space, \( L(H) \) the bounded linear operators on \( H \), \( B \) a unital \( C^* \)-algebra, and let \( A \) be a subalgebra of \( B \) containing the unit of \( B \). We call \( A \) an operator algebra. We let \( M_n \) denote the
Given a map $\rho: A \to L(H)$, we define maps $\rho_n: M_n(A) \to L(H + \cdots + H)$ ($n$ copies) by $\rho_n((a_{ij})) = (\rho(a_{ij}))$ for $(a_{ij})$ in $M_n(A)$. We call $\rho$ completely bounded provided that $\sup_n \|\rho_n\|$ is finite and we let $\|\rho\|_{cb}$ denote this supremum. If $\|\rho\|_{cb} \leq 1$, then we say that $\rho$ is completely contractive. By [1], a homomorphism $\rho$ of an operator algebra $A$ into $L(H)$ is completely contractive if and only if it can be dilated to $B$, that is, if and only if there exists a $*$-representation $\Pi: B \to L(K)$ for some Hilbert space $K$, containing $H$, such that $\rho(a) = P\Pi(a)\Pi I_H$ for all $a$ in $A$, where $P$ denotes the projection of $K$ onto $H$.

**THEOREM.** Let $A$ be an operator algebra contained in a $C^*$-algebra $B$, and let $\rho: A \to L(H)$ be a unital, completely bounded homomorphism. Then there exists an invertible operator $S$, with $\|S\| \cdot \|S^{-1}\| \leq \|\rho\|_{cb}$ such that $S^{-1}\rho(-)S$ is a completely contractive homomorphism.

**PROOF.** By the generalization of Stinespring’s Theorem [6, Theorem 2.8] and by [6, Theorem 2.4], there exists a Hilbert space $K$, a $*$-homomorphism $\Pi: B \to L(K)$, and two bounded operators $V_i: H \to K$, $i = 1, 2$, with $\|V_1\| \cdot \|V_2\| = \|\rho\|_{cb}$ such that $\rho(a) = V_1^*\Pi(a)V_2$, for $a$ in $A$.

Following [4, p. 1030], for $h \in H$, we define

$$\|h\| = \inf \left\{ \left\| \sum \Pi(a_i)V_2h_i \right\| : \sum \rho(a_i)h_i = h, \ a_i \in A, \ h_i \in H \right\},$$

where the infimum is taken over finite sums. By a minor modification of the arguments in [4, p. 1030], one obtains that $|\cdot|$ is a norm on $H$ and $(H, |\cdot|)$ is a Hilbert space.

If $h = \sum \rho(a_i)h_i$, then

$$\|h\| = \left\| \sum \rho(a_i)h_i \right\| = \left\| \sum V_1^*\Pi(a_i)V_2h_i \right\| \leq \|V_1^*\| \cdot \left\| \sum \Pi(a_i)V_2h_i \right\|.$$

Thus, $\|h\| \leq \|V_1^*\| \cdot |h|$. Similarly, $\rho(1)h = h$ yields $|h| \leq \|V_2\| \cdot \|h\|$. Thus, if we define $S: (H, |\cdot|) \to (H, \|\cdot\|)$ to be the identity, then $S$ is invertible and

$$\|S^{-1}\| \cdot \|S\| \leq \|V_1^*\| \cdot \|V_2\| = \|\rho\|_{cb}.$$

To complete the proof of the theorem it will be sufficient to prove that $S^{-1}\rho(-)S$ is completely contractive, since then $\|S^{-1}\| \cdot \|S\| \geq \|\rho\|_{cb}$ necessarily.

Let $a \in A$, and let $h = \sum \rho(a_i)h_i$. Then

$$|\rho(a)h| \leq \left\| \sum \Pi(aa_i)V_2h_i \right\| \leq \|a\| \cdot \left\| \sum \Pi(a_i)V_2h_i \right\|,$$

so $|\rho(a)h| \leq \|a\| \cdot |h|$. Thus, we obtain that $S^{-1}\rho(-)S$ is contractive.

To see that $S^{-1}\rho(-)S$ is completely contractive, fix an integer $n$, let $\hat{H} = H + \cdots + H$ ($n$ copies) and let $|\cdot|_n$ denote the norm on $\hat{H}$ given by $|\hat{h}|_n = |h_1|^2 + \cdots + |h_n|^2$, $\hat{h} = (h_1, \ldots, h_n)$. We must prove that if $\hat{a} = (a_{i,j}) \in M_n(A)$ then $|\rho_n(\hat{a})\hat{h}|_n \leq \|\hat{a}\| \cdot |\hat{h}|_n$ for $\hat{h} \in \hat{H}$.

To this end, consider $\rho_n: M_n(A) \to L(\hat{H})$ where $\hat{H}$ is endowed with its old norm, i.e., $\|\hat{h}\|^2 = |h_1|^2 + \cdots + |h_n|^2$. Since $\rho$ is completely bounded, $\rho_n$ will...
be completely bounded and, in fact, \( \|\rho_n\|_{cb} = \|\rho\|_{cb} \). Thus, by the first part of our argument we may endow \( \hat{H} \) with yet another norm \( |\cdot|_{(n)} \) such that \( \rho_n(\cdot) \) is contractive in this norm, i.e., \( |\rho_n(\hat{a})\hat{h}|_{(n)} \leq \|\hat{a}\| \cdot |\hat{h}|_{(n)} \).

To construct \( |\cdot|_{(n)} \), all we need is a Stinespring representation of \( \rho_n \). For such a representation, consider \( \Pi_n: M_n(B) \to L(\hat{K}) \), \( \hat{K} = K + \cdots + K \) (\( n \) copies) and \( \hat{V}_i: \hat{H} \to \hat{K} \) defined by \( \hat{V}_i(h_1, \ldots, h_n) = (V_i h_1, \ldots, V_i h_n) \), \( i = 1, 2 \). It is easily seen that \( \rho_n(\hat{a}) = \hat{V}_1 \Pi_n(\hat{a}) \hat{V}_2 \) for \( \hat{a} \in M_n(A) \). Thus, we may set \( |\hat{h}|_{(n)} = \inf \left\{ \left\| \sum_n \Pi_n(\hat{a}_i) \hat{V}_2 \hat{h}_i \right\| : \sum \rho_n(\hat{a}_i) \hat{h}_i = \hat{h} \} \), and \( \rho_n \) will be contractive in this norm.

We claim that with these choices \( |\hat{h}|_{(n)} = |\hat{h}|_n \), which will complete the proof of the theorem.

To prove the claim fix \( \varepsilon > 0 \), let \( \hat{a}_k = (a_{i,j,k}) \in M_n(A) \), \( \hat{h}_k = (h_{x,k}, \ldots, h_{n,k}) \in \hat{H} \) be such that, \( \sum \rho_n(\hat{a}_k) \hat{h}_k = \hat{h} \), and \( |\hat{h}|^2_{(n)} + \varepsilon \geq \left\| \sum_n \Pi_n(\hat{a}_k) \hat{V}_2 \hat{h}_k \right\|^2 \). We then have that
\[
|\hat{h}|^2_{(n)} + \varepsilon \geq \sum_{i=1}^n \left\| \sum_{j=1}^n \Pi_n(a_{i,j,k}) \hat{V}_2 \hat{h}_{j,k} \right\|^2 \geq \sum_{i=1}^n |\hat{h}_i|^2 = |\hat{h}|^2_n,
\]
and so \( |\hat{h}|_{(n)} \geq |\hat{h}|_n \). The other inequality follows similarly. This completes the proof of the theorem.

To see how Rota’s Theorem [7] follows from the above, let \( T \) be an operator whose spectrum is contained in the open unit disk. Recall that by the Riesz functional calculus, if \( f(z) \) is a polynomial, then
\[
f(T) = \frac{1}{2\pi i} \int_{\Gamma} f(z)(T - zI)^{-1} \, dz,
\]
where \( \Gamma = \{z: |z| = 1\} \). Setting \( \rho(f) = f(T) \), and letting \( \|f\| = \sup\{|f(z)|: |z| = 1\} \), we have that \( \|\rho(f)\| \leq K\|f\| \), where
\[
K = \frac{1}{2\pi} \int_{\Gamma} \|(T - zI)^{-1}\| \, d|z|.
\]

Thus, \( \rho \) extends to a bounded homomorphism of the disk algebra. To see that \( \rho \) is completely bounded (here we are thinking of the disk algebra as a subalgebra of the \( C^* \)-algebra of continuous functions on the circle), observe that for an \( n \times n \) matrix of polynomials,
\[
(f_{i,j}(T)) = \frac{1}{2\pi i} \int (f_{i,j}(z)(T - zI^{-1})) \, dz
\]
\[
= \frac{1}{2\pi i} \int (f_{i,j}(z)) (\hat{T} - z\hat{I}) \, dz,
\]
where \( \hat{T} \) is the direct sum of \( n \) copies of \( T \). Since \( \|(T - zI)^{-1}\| = \|\hat{T} - z\hat{I}\|^{-1} \),
we have
\[
\|(f_{i,j}(T))\| \leq K\|(f_{i,j}(z))\|,
\]
and so \( \rho \) is completely bounded with \( \|\rho\|_{cb} \leq K \). Hence, there is an invertible operator \( S \) such that \( \|S^{-1}\| \cdot \|S\| \leq K \) and \( \|S^{-1}TS\| = \|S^{-1}\rho(z)S\| \leq 1 \).
As a second application we mention the $\rho$-dilations considered in Sz.-Nagy and Foias [8]. An operator $T$ in $L(H)$ has a $\rho$-dilation if there is a unitary $U$ acting on $K$, $H$ contained in $K$, such that $T^n = \rho P U^n |_H$, $n \geq 1$, where $P$ is the projection of $K$ onto $H$. For $f$ in the disk algebra, define $\phi(f) = Pf(U)|_H$, and $\Psi(f) = f(0) \cdot I$. One easily sees that $\phi$ and $\Psi$ are complete contractions.

Finally, setting $\gamma(f) = f(T) = \rho \phi(f) + (1-\rho) \Psi(f)$, we have that $\gamma$ is a completely bounded homomorphism, and $\|\gamma\|_{cb} \leq 2\rho - 1$. Thus, there is an invertible $S$, $\|S^{-1}\| \cdot \|S\| \leq 2\rho - 1$, such that $S^{-1}TS$ is a contraction.

References

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