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# A NONCOMPLETELY CONTINUOUS OPERATOR ON $L_1(G)$ WHOSE RANDOM FOURIER TRANSFORM IS IN $c_0(\hat{G})$

N. GHOUSSOUB AND M. TALAGRAND

ABSTRACT. Let $T$ be a bounded linear operator from $L_1(G, \lambda)$ into $L_1(\Omega, \mathcal{F}, P)$, where $(G, \lambda)$ is a compact abelian metric group with its Haar measure, and $(\Omega, \mathcal{F}, P)$ is a probability space. Let $(\mu_\omega)$ be the random measure on $G$ associated to $T$; that is, $Tf(\omega) = \int_G f(t)\, d\mu_\omega(t)$ for each $f$ in $L_1(G)$.

We show that, unlike the ideals of representable and Kalton operators, there is no subideal $B$ of $\mathcal{M}(G)$ such that $T$ is completely continuous if and only if $\mu_\omega \in B$ for almost $\omega$ in $\Omega$. We actually exhibit a noncompletely continuous operator $T$ such that $\hat{\mu}_\omega \in l_{2+\varepsilon}(\hat{G})$ for each $\varepsilon > 0$.

**I. Random measures associated to operators on $L_1$.** Let $K$ be a separable compact Hausdorff space, $\lambda$ a Radon probability on $K$ and $(\Omega, \mathcal{F}, P)$ a probability space. Denote by $\mathcal{M}(K)$ the space of all Radon measures on $K$. We will call *random measure* on $K$ every measurable map $\omega \to \mu_\omega$ from $(\Omega, \mathcal{F}, P)$ into $\mathcal{M}(K)$ when $\mathcal{M}(K)$ is equipped with the $\sigma$-field generated by the $\sigma(\mathcal{M}(K), C(K))$ topology.

The starting point of this paper is the following disintegration theorem established by Kalton [**3**] and Fakhoury [**2**].

(a) If $T$ is a bounded linear operator from $L_1(K, \lambda)$ into $L_1(\Omega, \mathcal{F}, P)$, then there exists an essentially unique random measure $(\mu_\omega)$ verifying the following properties:

(i) Each $f$ in $L_1(K, \lambda)$ belongs to $L_1(K, |\mu_\omega|)$ for $P$ almost all $\omega$.

(ii) $Tf(\omega) = \int_K f(t)\, d\mu_\omega(t)$ for $P$ almost all $\omega$.

(iii) $\int |\mu_\omega|\, dP(\omega) \leq M \cdot \lambda$, where $M = \|T\|$.

(b) Conversely, every random measure $(\mu_\omega)$ verifying (i) and (iii) defines uniquely an operator $T$ from $L_1(K, \lambda)$ into $L_1(\Omega, \mathcal{F}, P)$ verifying (ii).

One can easily see that the lattice properties of the operators are compatible with the lattice properties of the random measures associated to them; that is, if $T = T^+ - T^-$ is the canonical decomposition of $T$ into its positive and negative parts, then $T^+$ and $T^-$ are given by the random measures $(\mu_\omega^+)$ and $(\mu_\omega^-)$ associated to $(\mu_\omega)$ by the Hahn-decomposition.

This representation allows us to associate to each order ideal $B$ in $\mathcal{M}(G)$ an order ideal of operators $\mathcal{L}_B(L_1(K), L_1(\Omega))$ in $\mathcal{L}(L_1, L_1)$ via the formula

$$T \in \mathcal{L}_B \text{ iff } \mu_\omega \in B \quad \text{for almost all } \omega.$$

In this paper, we investigate whether a reasonable order ideal of operators $V$ in $\mathcal{L}(L_1, L_1)$ can be characterized in terms of its associated random measures belonging in a suitable order ideal in $\mathcal{M}(K)$. A positive answer was given by Fakhoury [**2**] in the case of the space of *representable operators* (i.e. the operators $T$ such that there exists a Bochner integrable function $\varphi$: $K \to L_1(\Omega)$

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with $Tf = \int_K f \cdot \varphi(t)\, d\lambda(t)$ for all $f$ in $L_1(K)$. Fakhoury shows that $T$ is representable if and only if $\mu_\omega$ is absolutely continuous with respect to $\lambda$ for almost all $\omega \cdot (\mu_\omega \in \mathcal{M}_{\mathrm{ac}}(K))$. Moreover, Kalton [**3**] proved that an operator $T$ in $\mathcal{L}(L_1(K), L_1(\Omega))$ does not preserve a nontrivial projection band of $L_1(K)$ ($T$ is a non-*Kalton* operator) if and only if for almost all $\omega, \mu_\omega$ is a continuous measure on $K$ ($\mu_\omega \in \mathcal{M}_{\mathrm{c}}(K)$).

Several subideals of $\mathcal{M}(K)$ can be defined if one assumes that $K$ is a compact abelian group $G$. For instance, let $\mathcal{M}_{\mathrm{sc}}(G)$ be the space of *strongly continuous* measures on $G$: these are the bounded regular Borel measures $\mu$ on $G$ such that $|\mu|(g+H) = 0$ for all $g$ in $G$, and all closed subgroups $H$ of $G$ such that $G/H$ is infinite. Let $\mathcal{M}_0(G)$ be the ideal of measures whose Fourier transforms go to zero at infinity. Following Rosenthal [**4**], we also define $\mathcal{M}_\Lambda(G)$, the space of $\Lambda$-measures: these are the measures $\mu$ such that for each $\delta > 0$, there exists $p > 2$ such that $H = \{\gamma; \gamma \in \hat{G},\ |\hat{\mu}(\gamma)| > \delta\}$ is a $\Lambda_p$ set (i.e. the $L_p$ and the $L_1$ norms are equivalent on the linear span of the characters in $H$). Note that

$$\mathcal{M}_{\mathrm{ac}}(G) \subseteq \mathcal{M}_0(G) \subseteq \mathcal{M}_\Lambda(G) \subseteq \mathcal{M}_{\mathrm{sc}}(G) \subseteq \mathcal{M}_{\mathrm{c}}(G).$$

One can define the corresponding order ideals of operators on $L_1(G)$ via the above representation.

Recall that an operator $T: L_1(G) \to L_1(\Omega)$ is said to be a *completely continuous operator* or a *Dunford-Pettis operator* (resp. an *Enflo operator*) if $T(\mathrm{Ball}(L_\infty(G)))$ is norm compact (resp. if there exists a subspace $X$ of $L_1(G)$, isomorphic to $L_1$, such that $T\,|_X$ is an isomorphism). It is well known [**4**] that a convolution operator $T_\mu$ on $L_1(G)$ is Dunford-Pettis if and only if $\mu \in \mathcal{M}_0(G)$ and $T_\mu$ is a non-Enflo operator whenever $\mu \in \mathcal{M}_\Lambda(G)$. In the following, we shall prove that the random analogues to these properties do not hold, and that unlike the representable and the Kalton operators, there is no ideal in $\mathcal{M}(G)$ which can characterize the Dunford-Pettis and the non-Enflo operators.

We shall need the following fact [**1**]: If $(X_n)$ is a sequence of mean zero Bernoulli random variables on a probability space $(\Omega, \mathcal{F}, P)$ then

$$(*) \qquad P\left[\left|\sum_{i=1}^{m} X_i\right| > \lambda\right] < e^{-2\lambda^2/m} \quad \text{for each } 0 \le \lambda \le m/2.$$

LEMMA. *For each $m$, there exists a nonnegligible set $A_m$ in $\{-1,1\}^m$ such that $|\int_{A_m} \chi\, d\lambda| \le 2\sqrt{m}\cdot 2^{-m/2}$ for each character $\chi$ of $\{0,1\}^m$ which is not the identity.*

PROOF. Let $\{\chi_k; 1 \le k \le 2^m\}$ be an enumeration of the characters. For each $k$, write $\chi_k = \chi_{B_k^1} - \chi_{B_k^2}$, where $B_k^2$ is the complement of $B_k^1$, and let $\{x_j^i; 1 \le j \le 2^{m-1}\}$ be an enumeration for the elements of each set $B_k^i$ ($i = 1,2$).

Define now, on the probability space of the subsets of $\{0,1\}^m$ (which can be identified with $\{0,1\}^{2m}$), the following Bernoulli random variables:

$$X_{x_j}^i(A) = \begin{cases} 0 & \text{if } x_j \notin B_k^i \cap A, \\ 1 & \text{if } x_j \in B_k^i \cap A, \end{cases} \qquad i = 1,2.$$

We get from $(*)$ that

$$P\left(A; \left|\sum_{j=1}^{2^{m-1}} X_{x_j}^i(A) - 2^{m-2}\right| > \alpha 2^{(m-1)/2}\right) < e^{-2\alpha^2}, \qquad i = 1,2,$$

whenever $0 < \alpha \leq 2^{m-3/2}$. In other words if we denote by $\Omega_k^i$ the set

$$\{A; |\mathrm{card}(A \cap B_k^i) - 2^{m-2}| \leq \alpha 2^{m-1/2}\},$$

we have

$$P\left(\bigcap_{\substack{k=1\\ i=1,2}}^{2^m} \Omega_k^i\right) \geq 1 - \sum_{\substack{k=1\\ i=1,2}}^{2m} P([\Omega_k^i]) \geq 1 - 2^{m+1} \cdot e^{-2\alpha^2}.$$

If we take $\alpha = \sqrt{m}$, we get that there exists $A_m$, such that for each $\{B_k^i; i = 1,2,\ 1 \leq k \leq 2^m\}$,

$$(**) \qquad |\mathrm{card}(A_m \cap B_k^i) - 2^{m-2}| \leq \sqrt{m} \cdot 2^{m/2}.$$

Let $\chi_k = \chi_{B^1} - \chi_{B^2}$; we have

$$\begin{aligned}\left|\int_{A_m} \chi_k \, d\lambda\right| &= 2^{-m}\left[|\mathrm{card}(A_m \cap B_k^1) - \mathrm{card}(A_m \cap B_k^2)|\right]\\ &\leq 2 \cdot 2^{-m} \cdot \sqrt{m} \cdot 2^{m/2} = 2\sqrt{m} \cdot 2^{-m/2}.\end{aligned}$$

Now we can prove the following

THEOREM. *There exists a compact abelian group $G$ and an Enflo operator $T: L_1(G) \to L_1(\{0,1\}^{\mathbf{N}})$ such that the Fourier transform of the random measure associated to $T$ is in $l^{2+\varepsilon}$ for each $\varepsilon > 0$.*

PROOF. For each $m$, define the random measure $\mu^m: \{0,1\} \to \{-1,1\}^m$ by $\mu_0^m = \lambda_{A_m}/\lambda(A_m)$ and $\mu_1^m = \lambda_{A_m^c}/\lambda(A_m^c)$, where $\lambda_B$ denotes the restriction of the Haar measure on $\{-1,1\}^m$ on the set $B$.

Let now $G$ be the group $\prod_m \{-1,1\}^m$ and define the random measure $\mu$: $\{0,1\}^{\mathbf{N}} \to G$ by $\mu_{(x_1,x_2,\ldots)} = \bigotimes_m \mu_{x(m)}^m$.

Let $T$ be the operator from $L_1(G) \to L_1\{0,1\}^{\mathbf{N}}$, defined by

$$Tf(x_1,x_2,\ldots) = \int_G f(t)\, d\mu_{(x_1,x_2,\ldots)(t)}.$$

To prove that $T$ is bounded, it is enough to notice that for each set $A$ in $G$, we have

$$\int \mu_{(x_1,x_2,\ldots)}(A)\, d\lambda(x_1,x_2,\ldots) = \mu(A),$$

where $\lambda$ is the Haar measure on $\{0,1\}^{\mathbf{N}}$ and $\mu$ is the Haar measure on $G = \prod_m \{-1,1\}^m$. For each finite set $F \subseteq \mathbf{N}$, let $A_F = \{\chi; \chi = \prod_{k \in F} \chi^k$ and $\chi^k$ is a character on $\{-1,1\}^k$ which is different from one$\}$. Note that

$$\sum_{\chi \in A_F} |TX|^{2+\varepsilon} \leq 2^{L_F} \cdot \left(\prod_{k \in F} 2\sqrt{k} 2^{-k/2}\right)^{2+\varepsilon} \leq \prod_{k \in F} (4k)^{1+\varepsilon/2} \cdot 2^{-\varepsilon k/2},$$

where $L_F = \sum_{k \in F} k$.

Therefore, for each $q$,

$$\sum_{\text{card}(F)=q} \sum_{\chi \in A_F} |T\chi|^{2+\varepsilon} \leq \sum_{\text{card}(F)=q} \prod_{k \in F} (4k)^{1+\varepsilon/2} \cdot 2^{-\varepsilon k/2}$$

$$\leq (q!)^{-1} \left( \sum_k (4k)^{1+\varepsilon/2} \cdot 2^{-\varepsilon k/2} \right)^q .$$

Finally,

$$\sum_\chi |T\chi|^{2+\varepsilon} \leq \exp \left( \sum_k (4k)^{1+\varepsilon/2} \cdot 2^{-\varepsilon k/2} \right) < \infty \quad \text{for each } \varepsilon > 0.$$

Hence for each $x = (x_1, x_2, \ldots)$ in $\{0,1\}^{\mathbf{N}}$, $\hat{\mu}_x \in l_{2+\varepsilon}(\hat{G})$.

Now let $B_m = \{(y_n) \in \prod_n \{-1,1\}^n;\ y_m \in A_m\}$. Note that

$$TX_{B_m}(x) = \mu_{x_m}^m (A_m) = \begin{cases} 1 & \text{if } x_m = 0, \\ 0 & \text{if } x_m = 1, \end{cases}$$

from which follows that $T$ is not a Dunford-Pettis operator. Moreover, if $\mathcal{F}$ is the $\sigma$-field generated by $\{B_m; m \in \mathbf{N}\}$ and if we let $S$ be the operator $T$ restricted to $L_1[G, \mathcal{F}]$, one can easily see that the random measures $\nu_{(x_1, x_2, \ldots)}$ associated to $S$ are Dirac measures, which implies that $S$ is a Kalton operator and that $T$ is an Enflo operator.

COROLLARY. *There is no subset $B$ and $\mathcal{M}(G)$ such that the ideal of Dunford-Pettis operators (or the non-Enflo operators) is equal to $\mathcal{L}_B(L_1(G))$.*

PROOF. If there exists such a $B$ for the Dunford-Pettis operators (resp. non-Enflo operators) then by considering convolution operators we must have $\mathcal{M}_0(G) \subseteq B$ (resp. $\mathcal{M}_\Lambda(G) \subseteq B$). But the above example shows the existence of an Enflo operator $T$ such that the corresponding random measures belong to $\mathcal{M}_0(G)$ and hence to $B$, which is a contradiction.

REMARK. There exists a Dunford-Pettis operator on $L_1$ of a group $G$ such that none of the random measures associated to it belongs to $\mathcal{M}_0(G)$. For that, it is enough to take $G = \prod_n \{-1,1\}^{\{0,1\}^n}$ and, for each $s$ in $\{0,1\}^n$, let $A_s^n = \{y \in \{-1,1\}^{\{0,1\}^n}; y(s) = 1\}$ and $\nu_n^s = 2\mu_{A_s^n}^n$ (the restriction of the Haar measure $\mu^n$ of $\{-1,1\}^{\{0,1\}^n}$ on the set $A_s^n$). We leave it to the reader to check that the operator $T: L_1(G) \to L_1\{\prod_n \{0,1\}^n\}$ associated to the random measure $\nu: \prod_n \{0,1\}^n \to \mathcal{M}(G)$ defined by $\nu(x_1, x_2, \ldots) = \bigotimes_n \nu_n^{x_n}$ verifies the claimed properties.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER, BRITISH COLUMBIA, CANADA V6T 1W5

DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ PARIS VI, JUSSIEU, FRANCE