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COUNTEREXAMPLES TO SEVERAL PROBLEMS ON THE FACTORIZATION OF BOUNDED LINEAR OPERATORS

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ABSTRACT. For every $1 \leq p < \infty$, there exist a Banach lattice X_p and a lattice homomorphism T_p from X_p onto c_0 which satisfy:

(1) T_p does not preserve an isomorphic copy of c_0.

(2) T_p is a Radon-Nikodym operator.

(3) T_1 maps weakly Cauchy sequences into norm convergent sequences.

(4) If T_p is written as the product of two operators, then one of them preserves a copy of c_0.

0. Introduction. In [**4**], T. Figiel and the authors constructed the spaces X_p and the lattice homomorphisms T_p from X_p onto c_0 mentioned in the Abstract and verified (1). It turns out that these examples also can be used to answer several questions on the factorization of operators between Banach spaces of the following kind (see Pietsch's book [**13**] for a complete discussion of this type of problem): Given a property (P) for operators which determines a Banach ideal of operators, is it true that every operator with property (P) can be factored as the product of two operators with property (P)? In this note we show that the answer is negative for the properties (P) = Radon-Nikodym and (P) = does not preserve a copy of c_0, thus answering, respectively, questions of Stegall [**15**] and Pełczynski (see also Pietsch [**13**, Conjectures 2.6.2 and 3.1.9]).

Unexplained notation can be found in the books of Lindenstrauss and Tzafriri [**7**, **8**]. Here we note only that an operator T from a Banach space X is called a Radon-Nikodym operator if T takes X-valued bounded martingales into martingales which converge pointwise almost everywhere. An operator T from X preserves a copy of c_0 if there is a subspace of X isomorphic to c_0 such that the restriction of T to the subspace is an isomorphism.

We thank Michel Talagrand for several useful conversations. In particular, the result (2) that T_p is a Radon-Nikodym operator is due to Talagrand with essentially the proof that we present here.

I. The counterexamples. We now recall the space constructed in [**4**]. Let c be the space of converging sequences and set $X = l_1(c)$; that is, the space of doubly-indexed sequences $a = (a_{i,j})$, where $i = 1, 2, \ldots$; $j = 1, 2, \ldots, \omega$ such that

$$\lim_{j \to \infty} a_{i,j} = a_{i,\omega} \quad \text{for } i = 1, 2, \ldots$$

and

$$\|a\|_X = \sum_{i=1}^{\infty} \sup_j |a_{i,j}| < \infty.$$

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Define f_n in X by

$$(f_n)_{i,j} = \begin{cases} 1 & \text{if } i \leq n \leq j \leq \omega, \\ 0 & \text{otherwise.} \end{cases}$$

Let X_0 be the dense sublattice of X consisting of those vectors $a = (a_{i,j})$ whose rows are eventually zero; i.e., for some n, $a_{i,j} = 0$ for all $i \geq n$ and all $j = 1, 2, \ldots$.

On X_0 we define a new lattice norm by setting

$$\|x\|_1 = \inf\left\{ \|g\|_X + \sum_{i=1}^{n} |\alpha_i| : |x| \leq g + \sum_{i=1}^{n} \alpha_i f_i \right\}.$$

Let $(X_1, \| \ \|_1)$ be the completion of $(X_0, \| \ \|_1)$ and for $1 < p < \infty$, let $(X_p, \| \ \|_p)$ be the completion of the p-convexification of $(X_0, \| \ \|_1)$; that is, for $x \in X_0$

$$\|x\|_p = \| \, |x|^p \, \|_1^{1/p}.$$

Define an operator $T: X_0 \to c_0$ by setting $Ta = (a_{i,\omega})_{i=1}^{\infty}$. In [**4**] it was shown that for each $1 \leq p < \infty$, T extends to a norm one operator from the space X_p into c_0 (call the extension T_p).

The following properties were proved in [**4**].

(i) X_p contains no subspace isomorphic to l_1 for $p > 1$.

(ii) The operator T_p is a lattice homomorphism and a quotient map from X_p onto c_0, and T_p is strictly singular; in particular, T_p satisfies condition (1) in the Abstract.

We now verify condition (4) in the Abstract: First, observe that if x is any vector in X which is supported on a single row, then for each $1 \leq p < \infty$, $\|x\|_p = \|x\|_X$; i.e., the vectors in X_p supported on any given row form a Banach lattice which is isometrically isomorphic to c. Now suppose that $R: X_p \to Y$ and $S: Y \to c_0$ are bounded operators, $T_p = SR$, and R does not preserve a copy of c_0. It is well known (combine the remark after Lemma 1.a.5 with Propositions 2.a.1 and 1.a.12 in [**7**]) that an operator from c either preserves a copy of c_0 or is compact, so the above observation yields that the restriction of R to the vectors supported on the first n rows is a compact operator. Consequently, letting $x(n,i)$ be the indicator function of the set $\{(n,j): i \leq j \leq \omega\}$, we get that $Rx(n,j)$ converges to a limit, say, y_n, in Y as $j \to \infty$. Now $Sy_n = Tx(n,i)$ is the nth unit vector in c_0, so we have that for any scalars (α_n),

$$\left\|\sum \alpha_n y_n\right\| \geq \|S\|^{-1} \max |\alpha_n|.$$

Moreover, for any $i \geq n$ and any scalars $(\alpha_k)_{k=1}^{n}$, we see that

$$\left|\sum_{k=1}^{n} \alpha_k x(k,i)\right| \leq \max\{|\alpha_k| : 1 \leq k \leq n\} f_n,$$

and hence

$$\left\|\sum \alpha_n y_n\right\| \leq \|R\| \max |\alpha_n| \, \|f_n\|_p = \|R\| \max |\alpha_n|.$$

That is, the sequence (y_n) is equivalent to the unit vector basis of c_0 and the restriction of T_p to its closed linear span is an isomorphism. This completes the proof of (4).

In order to check the other properties of T_p, we need a technical proposition about X_p.

PROPOSITION 1. *Suppose that* $0 = k_0 < n_1 \leq k_1 < n_2 \leq k_2 < n_3 \leq \cdots$, $A_r = \{(i,j)\colon k_{r-1} < i \leq n_r \leq j \leq k_r\}$ *and* $d(r) = \operatorname{card} A_r$. *Let* E_r *be the vectors in* X *which are supported on* A_r. *Then* $[(E_r)_{r=1}^{\infty}]$ *is, in* X_p, *isometrically lattice isomorphic to* $(\sum_r l_\infty^{d(r)})_p$.

PROOF. First note that if x is a vector in X which is supported on a strip of the form

$$S_n = \{(i,j)\colon 1 \leq i \leq n \leq j \leq \omega\}$$

then for all $1 \leq p < \infty$, $\|x\|_p = \|x\|_\infty$. Indeed, it is easy to see (and was pointed out in [**4**]) that the evaluation functionals, defined for $1 \leq i < \infty$, $1 \leq j \leq \omega$ by $e(i,j)(x) = x_{i,j}$, are norm one functionals on each space X_p, hence $\|x\|_\infty \leq \|x\|_p$ for every x in X. But if x is supported on the strip S_n then $|x| \leq \|x\|_\infty f_n$ and $\|f_n\|_p = 1$, so $\|x\|_p \leq \|x\|_\infty$.

Thus for each r, E_r is lattice isometric to $l_\infty^{d(r)}$. Since X_p is p-convex, to complete the proof it is enough to check that if x is supported on $\bigcup_{r=1}^n E_r$, then

$$(*) \qquad \|x\|_p^p \geq \sum_{r=1}^{n} |1_{E_r} x|^p.$$

Since the E_r's are disjoint, it is enough to prove $(*)$ only for $p = 1$. This case is an immediate consequence of the following

CLAIM. *If* (i_r, j_r) *is in* E_r *for* $1 \leq r < \infty$, *then* $e(i_r, j_r)$ *is, in* X_1^*, *isometrically equivalent to the unit vector basis for* c_0.

To prove the Claim, note that

$$\left\|\sum a_r e(i_r, j_r)\right\|_{X_1^*} \geq \max |a_r|$$

because the $e(i_r, j_r)$'s are disjoint vectors in a Banach lattice. On the other hand, $i_r \neq i_s$ if $r \neq s$ so that

$$\left\|\sum a_r e(i_r, j_r)\right\|_{X^*} = \max |a_r|.$$

Moreover, for each n, (i_r, j_r) is in at most one strip $S_n = \operatorname{supp} f_n$, hence, for each $n = 1, 2, 3, \ldots$,

$$\sum |a_r| e(i_r, j_r)(f_n) \leq \max |a_r|.$$

This shows that

$$\left\|\sum a_r e(i_r, j_r)\right\|_{X_1^*} \leq \max |a_r|$$

and completes the proof of the Claim and Proposition 1.

Condition (3) in the Abstract is an immediate consequence of part (a) of Proposition 2. Moreover, Proposition 2 provides a proof of condition (1) in the Abstract which is probably simpler than that given in [**4**].

PROPOSITION 2. *Suppose that* (x_n) *is a bounded sequence in* X_p *and* $(T_p x_n)$ *is not relatively compact in* c_0.

(a) *If* $p = 1$, (x_n) *has a subsequence which is equivalent to the unit vector basis for* l_1 *and which spans a complemented subspace of* X_1.

(b) *If* $1 < p < \infty$, (x_n) *has a subsequence* (y_n) *such that* $(y_{2n} - y_{2n-1})$ *is equivalent to the unit vector basis for* l_p *and spans a complemented subspace of* X_p.

PROOF. There is no loss of generality in assuming that each x_n is in X_0. Since $(T_p x_n)$ is a bounded sequence in c_0 which is not relatively compact, we can assume by passing to a subsequence of the x_n's that

$$|(x_n)_{i(n),\omega}| > \mu > 0 \quad \text{with } i(n) \to \infty.$$

By passing to a further subsequence, we can pick $j(n)$ so that

$$|(x_n)_{i(n),j(n)}| > \mu \quad \text{and} \quad i(n) \leq j(n) < i(n+1).$$

Thus, by Proposition 1, the evaluation functionals $e(i(n), j(n))$ are, in X_p^*, isometrically equivalent to the unit vector basis for l_q $(1/p + 1/q = 1)$, and hence the operator $L\colon X_p \to l_p$ defined by

$$(Lx)(n) = e(i(n), j(n))(x)$$

is a norm one operator from X_p into l_p so that the nth coordinate of Lx_n is larger than μ. Therefore, in the case $p = 1$, by (an easy special case of) the results in [**14**], (Lx_n) has a subsequence which is equivalent to the unit vector basis of l_1 and which spans a complemented subspace of l_1. From this it is easy to check that the corresponding subsequence of (x_n) satisfies the conclusions of (a).

Since we do not use (b) in the sequel, we only sketch the idea for completing the proof in the case $p > 1$. First we pass to successive differences of a subsequence of the x_n's (call it (z_n)) so that (z_n) converges weakly to 0 in X_p and (Lz_n) is equivalent to the unit vector basis for l_p and spans a complemented subspace of L_p. This gives a lower l_p-estimate for linear combinations of the z_n's. Now we use the special form of X_p to justify that a subsequence of the z_n's is a small perturbation of a sequence of the form $(u_n + v_n)$, where the u_n's are supported on $\{(i,j)\colon j < \omega\}$ and where each of the bounded subsequences (u_n) and (v_n) consists of disjointly supported vectors. The upper l_p estimate for the corresponding subsequence of (z_n) then follows from the p-convexity of X_p.

We turn now to the verification of condition (2) in the Abstract. (This result is due to M. Talagrand.)

Given a bounded martingale $f^n\colon [0,1] \to X_p$, we need to show that $(T_p f^n)$ is almost surely norm convergent. It is well known that it is sufficient to check this only for the case that each f^n is a simple function, so we assume this extra condition.

Let Y be the weak*-limit in l_∞ of $(T_p f^n)$. It is also well known that if $(T_p f^n)$ is not almost surely norm convergent in c_0, then it is also not almost surely weakly convergent, so there is a subset C of $[0,1]$ with positive measure and a $\mu > 0$ so that for each t in C, $d(Y(t), c_0) > \mu$. By replacing C with a smaller subset of positive measure, we can assume that there is an increasing sequence (m_k) so that for each t in C,

(i) $\sup\{|Y(t)(m)|: m_k < m \leq m_{k+1}\} > \mu$.

Since for each coordinate m, the real valued martingale $(T_p f^n(m))$ converges to $Y(m)$, we can assume, by replacing C with a smaller set of positive measure, that there is an increasing sequence (n_k) so that for each t in C and $k = 1, 2, 3, \ldots,$

(ii) $\sup\{|T_p f^{n_k}(t)(m) - Y(t)(m)|: 1 \leq m \leq m_k\} < \mu/4$.

Since each f^n is a simple function, for each $k = 1, 2, 3, \ldots$, we can take $j_k > m_{k+1}$ so that for all t in C,

(iii) $\sup\{|f^{n_k}(t)(m, j_k) - T_p f^{n_k}(t)(m)|: 1 \leq m \leq m_{k+1}\} < \mu/4$.

Combining (i), (ii) and (iii) we get for all t in C that

$$\sup\{|f^{n_k}(t)(m, j_k)|: m_k < m \leq m_{k+1}\} < \mu/4.$$

Let $(k(r))$ be a sequence so that $m_{k(r+1)} > j_{k(r)}$ and set

$$A_r = \{(i, j_{k(r)}): m_{k(r)} < i \leq m_{k(r)+1}\}.$$

Let P_r (respectively P) be the band projection from X_p onto the functions supported on A_r (respectively on the union of all the A_r's). By Proposition 1, the range of P is an l_p-sum of finite-dimensional spaces and thus has the Radon-Nikodym property. Therefore, the martingale (Pf^n) is norm convergent almost surely to, say, f, and thus for almost all t, $\|P_r f(t)\| < \mu/4$ for large r, while for t in C and all r, $\|P_r f^{n_{k(r)}}(t)\| > \mu/4$, which is a contradiction.

REMARK. It was noted in [**4**] that for $p > 1$, T_p is an example of a nonweakly compact operator from a Banach lattice not containing a complemented copy of l_1 which does not preserve a copy of c_0. This showed that Pełczynski's theorem [**11**] (unlike some other results) that nonweakly compact operators from $C(K)$ spaces preserve a copy of c_0 does not generalize to operators from such lattices. On the other hand, it is shown in [**5**] that Grothendieck's [**6**] forerunner to Pełczynski's theorem does have a lattice analogue; in fact, it is proved that every operator from a Banach lattice not containing a complemented copy of l_1 into a Banach space not containing a copy of c_0 is weakly compact.

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