PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 92, Number 2, October 1984

ON THE SECOND DUAL OF THE LORENTZ SPACE

PRATIBHA G. GHATAGE AND BRIAN M. SCOTT

ABSTRACT. If $\phi(t) = t^{1/p}$ $(p > 1)$ and $(X, \mathcal{S}, \mu)$ is a completely nonatomic finite measure space, then the dual of the Lorentz space N_ϕ is denoted by M_ϕ and the closure of the simple functions in M_ϕ by M_ϕ^0. It is known that $(M_\phi^0)^* = N_\phi$. In this note we show that given a positive number $\beta < 1$ it is possible to construct a set of contractive embeddings of $(l_\infty/c_0)^*$ into $(M_\phi/M_\phi^0)^*$, each of which is bounded below by $M = M(\beta) \to 1$ as $\beta \to 0^+$. The union of the ranges of these embeddings is a total set in $(M_\phi/M_\phi^0)^*$.

In this note we study the bounded linear functionals on M_ϕ which annihilate the simple functions in M_ϕ. The dual space of M_ϕ^0 (= the closure of the simple functions in M_ϕ) has been shown to be isometrically isomorphic to N_ϕ, and thus the present results contribute toward a somewhat better understanding of M_ϕ^*.

NOTATION. Suppose that $(X, \mathcal{S}, \mu)$ is a completely nonatomic finite measure space, and $\phi(t) = t^{1/p}$ $(p > 1)$ is a concave function. Then the Lorentz space N_ϕ is defined by

$$N_\phi = \left\{ f\colon\ f \text{ is measurable \& } \int_0^\infty \phi(\mu(F_y))\, dy < \infty \right\},$$

where $F_y = \{x \in X\colon\ |f(x)| > y\}$;

$$M_\phi = \left\{ f\colon\ f \text{ is measurable \& } \sup_{\mu(E)>0} \frac{1}{\phi(\mu(E))} \int_E |f(x)|\, d\mu < \infty \right\}$$

and

$$M_\phi^0 = \left\{ f \in M_\phi\colon\ \lim_{\mu(E)\to 0} \frac{1}{\phi(\mu(E))} \int_E |f(x)|\, d\mu = 0 \right\}.$$

It is known that $N_\phi^* = M_\phi$, M_ϕ^0 is the closure of the simple functions in M_ϕ, and $(M_\phi^0)^* = N_\phi$. For more details see [**2** and **3**]. Finally, q is defined as usual by $1/p + 1/q = 1$.

REMARK. It is easy to verify that N_ϕ is an (AL)-space, since the simple functions are dense in N_ϕ; M_ϕ is therefore isometrically isomorphic to $C(S)$, where the compact Hausdorff space S may be taken to be the extreme points of the positive face of the unit sphere of M_ϕ^* with the weak* topology. However, this abstract characterization seems not to help in finding a concrete description of a single element of $(M_\phi/M_\phi^0)^*$. (See [**1**, Chapter 6, §§1, 2].) We have instead the following construction.

Fix a positive number $\beta < 1$. The nonatomicity of μ allows us to choose a sequence $\mathcal{E} = \{E_n\}$ of pairwise disjoint, measurable sets with $\mu(E_n) = \beta^{np}$ (provided,

Received by the editors September 15, 1983.
1980 *Mathematics Subject Classification*. Primary 46E30.
Key words and phrases. Lorentz space.

of course, that $\sum_{n=1}^{\infty}\beta^{np}=\beta^p/(1-\beta^p)\leq\mu(X)$). We denote the set of all such sequences by $\mathbf{E}_\beta$.

LEMMA 1. *For each* $\mathcal{E}\in\mathbf{E}_\beta$ *define* $T_{\mathcal{E}}\colon (l_\infty/c_0)^*\to(M_\phi/M_\phi^0)^*$ *by* $T_{\mathcal{E}}(L)=\Lambda$, *where*

$$\Lambda(g+M_\phi^0)=L\left\{\frac{1}{\mu(E_n)^{1/p}}\int_{E_n}g\,d\mu+c_0\right\};$$

then T_E *is a contractive embedding.*

PROOF. Note first that for each $g\in M_\phi$,

$$\left\|\left\{\frac{1}{\mu(E_n)^{1/p}}\int_{E_n}g\,d\mu\right\}\right\|_\infty\leq\|g\|_{M_\phi}.$$

Moreover,

$$\left\{\frac{1}{\mu(E_n)^{1/p}}\int_{E_n}g\,d\mu\right\}\in c_0\quad\text{whenever } g\in M_\phi^0.$$

We now prove that $T_{\mathcal{E}}$ is bounded below by a number $M(\beta)$ that (i) is independent of E, and (ii) approaches 1 as $\beta\to 0^+$.

For any $\{r_n\}\in l_\infty$, $g=\sum(r_n/\mu(E_n)^{1/q})\psi_{E_n}$ is a well-defined function. To prove that $g\in M_\phi$ we need only look at

$$\frac{1}{\mu(E)^{1/p}}\int_E|g|\,d\mu\quad\text{for } E\subseteq\bigcup_n E_n.$$

If we write $\mu(E\cap E_n)=\alpha_n\mu(E_n)$, then $0\leq\alpha_n\leq 1$, and

$$\begin{aligned}\frac{1}{\mu(E)^{1/p}}\int_E|g|\,d\mu&=\frac{1}{(\sum\alpha_n\beta^{np})^{1/p}}\sum\frac{|r_n|\mu(E\cap E_n)}{\mu(E_n)^{1/q}}\\&=\frac{1}{(\sum\alpha_n\beta^{np})^{1/p}}\sum|r_n|\alpha_n\mu(E_n)^{1/p}\\&=\sum\frac{|r_n|\alpha_n\beta^n}{(\sum\alpha_n\beta^{np})^{1/p}}\leq\|\{r_n\}\|_\infty\frac{\sum\alpha_n\beta^n}{(\sum\alpha_n\beta^{np})^{1/p}}.\end{aligned}$$

Let

$$M(\beta)=\sup_{\substack{\{\alpha_n\}\\0\leq\alpha_n\leq1}}\frac{\sum\alpha_n\beta^n}{(\sum\alpha_n\beta^{np})^{1/p}}.$$

Then $\|g\|_{M_\phi}\leq M(\beta)\|\{r_n\}\|_\infty$, and the desired result follows from the next lemma.

LEMMA 2. $M(\beta)=(1-\beta^p)^{1/p}/(1-\beta)$.

PROOF. We start by proving a slightly more general result. Let $\{\beta_n\}$ be a decreasing sequence of positive numbers such that $\sum\beta_n<\infty$. Let $D=[0,1]^\infty\setminus\{\vec{0}\}$, where $\vec{0}=\langle 0,0,\ldots\rangle$. For $\vec{\alpha}=\{\alpha_n\}\in D$ define

$$f(\vec{\alpha})=\sum\beta_n\alpha_n\Big/\left(\sum\beta_n^p\alpha_n\right)^{1/p}.$$

Finally, for $k\geq 1$ define $\vec{z^k}\in D$ by

$$z_i^k=\begin{cases}0, & i<k,\\ 1, & i\geq k.\end{cases}$$

Our result will follow easily once we establish the following important claim.

Claim 1. $\sup_{\vec{\alpha}\in D} f(\vec{\alpha}) = \sup_k f(\vec{z^k})$.

To prove the claim we need some extra notation. For $\vec{\alpha}\in D$ and $n \geq 1$ let

$$t_n(\vec{\alpha}) = \sum_{k=1}^{n} \beta_k \alpha_k \quad \text{and} \quad b_n(\vec{\alpha}) = \left(\sum_{k=1}^{n} \beta_k^p \alpha_k\right)^{1/p}.$$

If $b_n(\vec{\alpha}) \neq 0$ let $f_n(\vec{\alpha}) = t_n(\vec{\alpha})/b_n(\vec{\alpha})$; clearly $f_n(\vec{\alpha}) \to f(\vec{\alpha})$ as $n \to \infty$. It suffices, therefore, to prove for any $\vec{\alpha}\in D$ and $n \geq 1$ that there is a $k \leq n$ such that $f_n(\vec{\alpha}) \leq f_n(\vec{z^k}) \leq f(\vec{z^k})$. The existence of such a k depends on the following elementary observations.

Fact 1. Suppose $0 < b \leq t$ and $\beta > 0$, and let $h(x) = (t + \beta x)/(b^p + \beta^p x)^{1/p}$ for $0 \leq x \leq 1$. Then $h(x)$ attains its maximum at $x = 0$ or $x = 1$. Moreover, $h(x)$ attains its maximum at $x = 1$ whenever $\beta^{p-1} \leq b^p/t$.

Fact 2. If $0 < x < b \leq t$ and $(t + x)/(b^p + x^p)^{1/p} < t/b$, then

$$(t - x)/(b^p - x^p)^{1/p} > t/b.$$

Fact 3. If $0 < x \leq y < b \leq t$ and $(t + x)/(b^p + x^p)^{1/p} < t/b$, then

$$(t - y)/(b^p - y^p)^{1-p} > t/b.$$

The proof of Fact 1 is trivial. If the conclusion of Fact 2 fails, then

$$(1 + x/t)^p < 1 + (x/b)^p \quad \text{and} \quad (1 - x/t)^p \leq 1 - (x/b)^p,$$

so

$$(1 + x/t)^p + (1 - x/t)^p < 2.$$

But $g(y) = (1+y)^p+(1-y)^p$ is an increasing function for $0 \leq y \leq 1$, and $g(0) = 2$, so, in particular, $g(x/t) > 2$; this contradiction establishes the result. Fact 3 generalizes and follows from Fact 2. To see this, note that as a function of x, $(t+x)/(b^p+x^p)^{1/p}$ is increasing if $0 \leq x < (b^p/t)^{1/(p-1)}$, decreasing if $x > (b^p/t)^{1/(p-1)}$, and equal to t/b when $x = 0$. The hypothesis of Fact 3 therefore implies that $x > (b^p/t)^{1/(p-1)}$, so the conclusion is immediate from Fact 2.

We can now prove Claim 1.

Fix $\vec{\alpha}\in D$ and $n \geq 1$. Let $\vec{\alpha^1}\in D$ be obtained from $\vec{\alpha}$ by changing α_n to 1. Since

$$\beta_n^{p-1} < \beta_{n-1}^{p-1} \leq \frac{(b_{n-1}(\vec{\alpha}))^p}{t_{n-1}(\vec{\alpha})} \leq 1,$$

Fact 1 implies that $f_n(\vec{\alpha^1}) \geq f_n(\vec{\alpha})$. Now replace α^1_{n-1} by 1 or 0 to get $\vec{\alpha^2}$, whichever makes $f_n(\vec{\alpha^2})$ larger; Fact 1 shows that $f_n(\vec{\alpha^2}) \geq f_n(\vec{\alpha^1})$. Continue in this way to obtain $\vec{\alpha^n}\in D$ such that $f_n(\vec{\alpha^n}) \geq f_n(\vec{\alpha})$, $\alpha^n_1, \alpha^n_2, \ldots, \alpha^n_n \in \{0, 1\}$, and $\alpha^n_n = 1$.

Suppose there are m and l such that $1 \leq m < l < n$, $\alpha^n_m = 1$, $\alpha^n_l = 0$, and $\alpha^n_{l+1} = \cdots = \alpha^n_n = 1$. Let $\vec{u}$ be obtained from $\vec{\alpha^n}$ by replacing α^n_m by 0, let $\vec{v}$ be obtained from $\vec{\alpha^n}$ by replacing α^n_l by 1, and assume that $f_n(\vec{\alpha^n}) > f_n(\vec{v})$.

It follows from Fact 3 that $f_n(\vec{u}) > f_n(\vec{\alpha^n})$: just set $t = t_n(\vec{\alpha^n})$, $b = b_n(\vec{\alpha^n})$, $x = \beta_l$, and $y = \beta_m$. In short, we may replace $\vec{\alpha^n}$ by either $\vec{u}$ or $\vec{v}$ without decreasing the value of f_n. It now follows by an easy finite induction that there is some $k \leq l$ such that $f_n(\vec{\alpha^n}) \leq f_n(\vec{z^k})$; and repeated applications of Fact 1, $f_n(\vec{z^k}) \leq f_{n+1}(\vec{z^k}) \leq f_{n+2}(\vec{z^k}) \leq \cdots$, whence $f_n(\vec{z^k}) \leq f(\vec{z^k})$, and the proof of Claim 1 is complete.

To finish the proof of Lemma 2, just note that if $0 < \beta < 1$ and $\beta_n = \beta^n$ for $n \geq 1$, then

$$f(\vec{z^k}) = \frac{\sum_{n=k}^{\infty} \beta^n}{(\sum_{n=k}^{\infty} \beta^{np})^{1/p}} = \frac{\beta^k/(1-\beta)}{[\beta^{kp}/(1-\beta^p)]^{1/p}} = \frac{(1-\beta^p)^{1/p}}{1-\beta},$$

and hence,

$$\sup_{\vec{\alpha} \in D} f(\vec{\alpha}) = \frac{(1-\beta^p)^{1/p}}{1-\beta}.$$

This is an increasing function of β, so for any $M > 1$ we can choose β to make $M(\beta) = M$. This completes the proofs of Lemmas 1 and 2.

LEMMA 3. *For any $M > 1$ there is a β, $0 < \beta < 1$, such that the family $\{T_{\mathcal{E}}\colon\ \mathcal{E} \in \mathbf{E}_\beta\}$ of embeddings of $(l_\infty/c_0)^*$ into $(M_\phi/M_\phi^0)^*$ satisfies the following condition: If $L \in (l_\infty/c_0)^*$, $\mathcal{E} \in \mathbf{E}_\beta$, and $\Lambda = T_{\mathcal{E}}(L)$, then $\|\Lambda\| \leq \|L\| \leq M\|\Lambda\|$.*

PROOF. Note that

$$\{r_n\} \mapsto g = \sum_{n=1}^{\infty} \frac{r_n}{\mu(E_n)^{1/q}} \chi_{E_n}$$

defines a bounded map from l_∞ into M_ϕ. Moreover, if $r_n \to 0$, then

$$g_n = \sum_{k=1}^{n} \frac{r_k}{\mu(E_k)^{1/q}} \chi_{E_k} \to g \quad \text{in the } M_\phi\text{-norm},$$

since $\|g - g_n\|_{M_\phi} \leq M\|\{r_k\}_{k=n}^{\infty}\|_\infty$; thus, $g \in M_\phi^0$. (Of course, β is chosen so that $M(\beta) = M$.) Hence the map

$$j_{\mathcal{E}}\colon\ l_\infty/c_0 \to M_\phi/M_\phi^0\colon\ \{r_n\} + c_0 \mapsto g + M_\phi^0$$

is well defined and $\|j_{\mathcal{E}}\| \leq M$. Also, if $j_{\mathcal{E}}(\{r_n\}) = g$, then

$$\frac{1}{\mu(E_n)^{1/q}} \int_{E_n} g\, d\mu = r_n;$$

thus, the adjoint $j_{\mathcal{E}}^*\colon\ (M_\phi/M_\phi^0)^* \to (l_\infty/c_0)^*$ satisfies $j_{\mathcal{E}}^* \circ T_{\mathcal{E}} = \mathrm{id}$ and $\|j_{\mathcal{E}}^*\| \leq M$. This completes the proof of Lemma 3.

We now prove a result about elements of $M_\phi \setminus M_\phi^0$; this will help us prove that the union of the ranges of all the $T_{\mathcal{E}}$ $(\mathcal{E} \in \mathbf{E}_\beta)$ is a total set in $(M_\phi/M_\phi^0)^*$.

LEMMA 4. *If $g \in M_\phi \setminus M_\phi^0$, there is a sequence $\{A_n\}$ of pairwise disjoint, measurable sets such that $\mu(A_n) \to 0$, but*

$$\inf_{n \geq 1} \frac{1}{\mu(A_n)^{1/p}} \int_{A_n} |g|\, d\mu > 0.$$

PROOF. Recall that

$$M_\phi^0 = \left\{ g \in M_\phi\colon \lim_{\mu(E)\to 0} \frac{1}{\mu(E)^{1/p}} \int_E |g|\, d\mu = 0 \right\}.$$

Thus if $g \notin M_\phi^0$, we may assume $g \geq 0$ and there is a sequence $\{A_n\}$ of measurable sets of pairwise measure such that if $\alpha_n = \mu(A_n)$, then $\lim_n \alpha_n = 0$, but $\inf_n a_n = \varepsilon > 0$, where $a_n = \alpha_n^{-1/p} \int_{A_n} g\, d\mu$.

Let

$$A_{n,m} = A_n \setminus \bigcup_{k\geq m} A_k, \quad \alpha_{n,m} = \mu(A_{n,m}), \quad a_{n,m} = \frac{1}{\alpha_{n,m}^{1/p}} \int_{A_{n,m}} g\, d\mu,$$

$$B_{n,m} = A_n \setminus A_{n,m}, \quad \beta_{n,m} = \mu(B_{n,m}), \quad \text{and} \quad b_{n,m} = \frac{1}{\beta_{n,m}^{1/p}} \int_{B_{n,m}} g\, d\mu.$$

Then

$$\begin{aligned} \varepsilon \leq a_n &= \alpha_n^{-1/p} \left(\int_{A_{n,m}} g\, d\mu + \int_{B_{n,m}} g\, d\mu \right) \\ &= \alpha_n^{-1/p} (\alpha_{n,m}^{1/p} a_{n,m} + \beta_{n,m}^{1/p} b_{n,m}) \leq \alpha_n^{-1/p} (\alpha_{n,m}^{1/p} a_{n,m} + \beta_{n,m}^{1/p} \|g\|_{M_\phi}). \end{aligned}$$

Hence,

$$\alpha_n^{1/p}\varepsilon - \beta_{n,m}^{1/p} \|g\|_{M_\phi} \leq \alpha_{n,m}^{1/p} a_{n,m} \quad \text{or} \quad a_{n,m} \geq \left(\frac{\alpha_n}{\alpha_{n,m}}\right)^{1/p} \varepsilon - \left(\frac{\beta_{n,m}}{\alpha_{n,m}}\right)^{1/p} \|g\|_{M_\phi}.$$

By passing, if necessary, to a subsequence, we may assume $\sum \alpha_n < \infty$, in which case $\lim_m(\beta_{n,m}/\alpha_{n,m}) = 0$, so $a_{n,m} \geq \varepsilon/2$ for sufficiently large m (and a fixed n).

For each n let $m(n)$ be such an m. Let $A_1' = A_{1,m(1)}$ and $n_1 = 1$. Given n_k, let $A_k' = A_{n_k, m(n_k)}$ and $n_{k+1} = m(n_k)$. Let $\alpha_k' = \mu(A_k')$ and $a_k' = \alpha_k^{-1/p} \int_{A_k'} g\, d\mu$. Clearly $\lim_k \alpha_k' = 0$, $\inf_k a_k' \geq \varepsilon/2$, and the sets A_k' are pairwise disjoint.

THEOREM 1. *The subset of $(M_\phi/M_\phi^0)^*$ consisting of the union of the ranges of all the embeddings $T_{\mathcal{E}}$ $(\mathcal{E} \in \mathbf{E}_\beta)$ is total.*

PROOF. Suppose $g \in M_\phi$ is such that whenever $L \in l_\infty^* \cap c_0^\perp$, $\mathcal{E} \in \mathbf{E}_\beta$, and $\Lambda = T_{\mathcal{E}}(L)$, then $\Lambda(g) = 0$. We show that $g \in M_\phi^0$.

Suppose not. Without loss of generality we may assume $g \geq 0$. By Lemma 4 there is a sequence $\{A_n\}$ of pairwise disjoint sets of positive measure such that if $\alpha_n = \mu(A_n)$ and $a_n = \alpha_n^{-1/p} \int_{A_n} g\, d\mu$, then $\lim_n \alpha_n = 0$ and $\inf_n a_n = \varepsilon > 0$. Clearly, we may further assume that $\alpha_1 > \alpha_2 > \cdots$, and indeed that for each n there is a $k(n)$ such that (i) $\beta^{pk(n)} > \alpha_n > \beta^{p(k(n)+1)}$, and (ii) $n < m$ implies $k(n) < k(m)$. (Pass to a subsequence if necessary.) Once again because μ is completely nonatomic we may choose sets B_n, disjoint from one another and from the A_n's so that $\mu(B_n) = \beta^{pk(n)} - \alpha_n$; let $E_n = A_n \cup B_n$. Then $\mu(E_n) = \beta^{pk(n)}$, and $\{E_n\}$ is a subsequence of some $\mathcal{E} \in \mathbf{E}_\beta$. Moreover, if $r_n = \beta^{-k(n)} \int_{E_n} g\, d\mu$, then $\{r_n\} \in l_\infty$. We now claim that $\{r_n\} \in c_0$.

By hypothesis, $L \in l_\infty^* \cap c_0^\perp$ implies $T(L)(g) = 0$ and, hence, $L\{s_n\} = 0$ for some sequence $\{s_n\}$ of which $\{r_n\}$ is a subsequence. But then $\{s_n\} \in c_0$, so $\{r_n\} \in c_0$ also.

But

$$\beta^{-k(n)} \int_{E_n} g\, d\mu = \beta^{-k(n)} \left[\int_{A_n} g\, d\mu + \int_{B_n} g\, d\mu \right] = \beta^{-k(n)} \left[\alpha_n^{1/p} a_n + \int_{B_n} g\, d\mu \right]$$
$$\geq \beta^{-k(n)} \alpha_n^{1/p} a_n > \frac{\beta^{k(n)+1}}{\beta^{k(n)}} a_n \geq \beta\varepsilon,$$

so $\{r_n\} \notin c_0$. This contradiction shows that $g \in M_\phi^0$ and completes the proof of Theorem 1.

Theorem 1 leaves open the question of how much of $(M_\phi/M_\phi^0)^*$ can be filled up with the ranges of the embeddings $T_{\mathcal{E}}$; we do not know the answer even if β is allowed to range over $(0,1)$. For a fixed β, at least, we suspect there are bounded linear functionals on M_ϕ/M_ϕ^0 that do not arise as linear combinations of functionals $T_{\mathcal{E}}(L)$, $\mathcal{E} \in \mathbf{E}_\beta$ and $L \in (l_\infty/c_0)^*$.

We do know that if $\mathcal{E} = \{E_n\}$ and $\mathcal{F} = \{F_n\}$ in $\mathbf{E}_\beta$ are eventually disjoint ($\bigcup_{n=k}^\infty E_n \cap \bigcup_{n=k}^\infty F_n = \varnothing$ for some $k \geq 1$), then the ranges of $T_{\mathcal{E}}$ and $T_{\mathcal{F}}$ make a positive angle (depending on β but not on $\mathcal{E}$ and $\mathcal{F}$) with each other. (In fact, a somewhat weaker hypothesis will suffice.) The idea is to construct a positive $g \in M_\phi$ that is annihilated by every functional in the range of $T_{\mathcal{F}}$ (because $\{\mu(F_n)^{-1/p} \int_{F_n} g\, d\mu\} \in c_0$) but is such that $\{\mu(E_n)^{-1/p} \int_{E_n} g\, d\mu\}$ converges to 1.

References

1. M. M. Day, *Normed linear spaces*, Springer-Verlag, New York, 1973.
2. P. Ghatage, *The Lorentz space as a dual space*, preprint.
3. M. S. Steigerwalt and A. J. White, *Some function spaces related to L_p spaces*, Proc. London Math. Soc. (3) **22** (1971), 137–163.

Department of Mathematics, Cleveland State University, Cleveland, Ohio 44115