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ON SUMMABILITY OF FOURIER SERIES AT A POINT

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ABSTRACT. In this paper summability of Fourier series by a regular linear method of summation determined by a triangular matrix, has been studied and various results—some known and some new—on Cesàro and Nörlund summability have been deduced. A convergence criterion has also been obtained.

1. Let $C = (c_{n,k})$, $k = 0, 1, 2, \ldots, n$, be a triangular matrix and let

$$t_n = \sum_{k=0}^{n} c_{n,k} s_k,$$

where $\{s_k\}$ is a given sequence of numbers. If $t_n \to s$ as $n \to \infty$, $\{s_n\}$ is called summable (C) to s. In this paper we assume $c_{n,k} \geq 0$ for $k = 0, 1, 2, \ldots, n$, and $\sum_{k=0}^{n} c_{n,k} = 1$. Then a necessary and sufficient condition for regularity of the method (C) is

$$\lim_{n\to\infty} c_{n,k} = 0 \quad \text{for each k.}$$

In the case

$$c_{n,k} = A_{n-k}^{\alpha-1}/A_n^{\alpha}, \qquad \alpha \geq 0,$$

where $\{A_n^{\alpha-1}\}$ is determined by the identity

$$(1-x)^{-\alpha} = \sum_{0}^{\infty} A_n^{\alpha-1} x^n \qquad (|x| < 1),$$

the method (C) reduces to the well-known Cesàro method (C, α). For

$$c_{n,k} = p_{n-k}/P_n, \qquad P_n = p_0 = p_1 + \cdots + p_n > 0,$$

the method (C) reduces to the Nörlund method (N, p). In the case $p_n = 1/(n+1)$, the Nörlund method $(N, 1/(n+1))$ is also known as the harmonic method.

Let f be a Lebesgue integrable periodic function with period 2π and let

$$f(x) \sim \frac{1}{2}a_0 + \sum_{1}^{\infty}(a_n \cos nx + b_n \sin nx) \equiv \sum_{0}^{\infty} A_n(x).$$

We write

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\},$$

$$\Phi(t) = \int_0^t |\phi(u)|\, du \quad \text{and} \quad s_n(x) = \sum_{0}^{n} A_k(x).$$

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Let

$$C_n(k) = \sum_{m=0}^{k} c_{n,n-m}$$

and, for $u \geq 0$, define $C_n(u) = C_n([u])$, where $[u]$ is the greatest integer function.

Throughout the paper K is used to denote an absolute constant, not necessarily the same at each occurrence.

2. We establish the following

THEOREM. *Let $\{c_{n,k}\}$ be nondecreasing with respect to k. Let χ be a positive function defined over $(0, \infty)$ such that as $n \to \infty$, (i) $n\chi(n) = O(1)$ and (ii) $\int_1^n \chi(u)C_n(u)\,du = O(1)$. Then if $\Phi(t) = o(\chi(\pi/t))$, as $t \to 0+$, the series $\sum A_n(x)$ is summable (C) to $f(x)$.*

3. Proof. We have that $\{c_{n,k}\}$ is nonnegative and nondecreasing in k. Hence,

$$(n-k)c_{n,k} \leq \sum_{m=k+1}^{n} c_{n,m} \leq 1.$$

Thus for each fixed k, $c_{n,k} \to 0$ as $n \to \infty$, that is, (C) is a regular method.

In view of the fact that the convergence of Fourier series at a point is a local property of the generating function, we may take $\phi(t) = 0$ over $[\delta, \pi]$, where $0 < \delta < \pi$. We choose δ such that $\Phi(t) = o(\chi(\pi/t))$ for $t \in (0, \delta)$. Let

$$t_n(x) = \sum_{0}^{n} c_{n,k}s_k(x).$$

Then we need to show that $t_n(x) - f(x) = o(1)$ as $n \to \infty$. After the Dirichlet integral, for $n > \pi/\delta$,

$$\begin{aligned} t_n(x) - f(x) &= \sum_{0}^{n} c_{n,k}s_k(x) - f(x) = \frac{1}{\pi}\int_0^{\delta} \phi(t)L(n,t)\,dt \\ &= \frac{1}{\pi}\left\{\int_0^{\pi/n} + \int_{\pi/n}^{\delta}\right\} = \mathrm{I}_1 + \mathrm{I}_2, \quad \text{say,} \end{aligned}$$

where

$$L(n,t) = \sum_{0}^{n} \frac{c_{n,k}\sin\left(k+\frac{1}{2}\right)t}{\sin\left(\frac{1}{2}t\right)}.$$

As

$$|L(n,t)| \leq \pi \sum_{0}^{n} \left(k+\tfrac{1}{2}\right)c_{n,k} \leq \pi\left(n+\tfrac{1}{2}\right),$$

we get

$$|\mathrm{I}_1| \leq \left(n+\frac{1}{2}\right)\int_0^{\pi/n} |\phi(t)|\,dt = o(n\chi(n)) = o(1),$$

as $n \to \infty$.

Next, in view of the order estimates of McFadden [**4**, Lemma 5.11],

$$\left|\sum_{k=a}^{b} c_{n,n-k}e^{i(n-k)t}\right| \leq KC_n(\pi/t),$$

where $0 \leq a \leq b \leq \infty$, $0 < t \leq \pi$, and n a positive integer, we obtain

$$\begin{aligned}|\mathrm{I}_2| &\leq K \int_{\pi/n}^{\delta} \frac{|\phi(t)| C_n(\pi/t)}{t}\, dt \\ &= K \sum_{k=r}^{n-1} \int_{\pi/(k+1)}^{\pi/k} \frac{|\phi(t)| C_n(\pi/t)}{t}\, dt + K \int_{\pi/r}^{\delta} \frac{|\phi(t)| C_n(\pi/t)}{t}\, dt,\end{aligned}$$

where r is a positive integer such that $\pi/r \leq \delta < \pi/(r+1)$. As

$$\int_{\pi/(k+1)}^{\pi/k} \frac{|\phi(t)| C_n(\pi/t)}{t}\, dt = \left[\frac{C_n(\pi/t)}{t}\Phi(t)\right]_{\pi/(k+1)}^{\pi/k} + \int_{\pi/(k+1)}^{\pi/k} \frac{\Phi(t) C_n(\pi/t)}{t^2}\, dt,$$

$$\begin{aligned}|\mathrm{I}_2| &\leq o(C_n(r)) + o(n\chi(n)C_n(n)) + K \int_{\pi/n}^{\delta} \frac{\Phi(t) C_n(\pi/t)}{t^2}\, dt \\ &= o(1) + o\left(\int_1^n \chi(u) C_n(u)\, du\right) = o(1).\end{aligned}$$

This completes the proof of the Theorem.

4. The four corollaries in this section follow as a result of our Theorem.

COROLLARY 1 (HARDY [**2**]). *Let $\alpha > 0$. If $\Phi(t) = o(t)$, as $t \to 0+$, then $\sum A_n(x)$ is summable (C, α) to $f(x)$.*

The case $\alpha = 1$ is the classical result of Lebesgue (see [**10**, Theorem III 3.9]).

PROOF. Let $\chi(u) = \pi/u$ and $c_{n,k} = A_{n-k}^{\alpha-1}/A_n^{\alpha}$. Then $\chi(\pi/t) = t$ and

$$C_n(u) = \sum_{m=0}^{[u]} c_{n,n-m} = \sum_{m=0}^{[u]} \frac{A_m^{\alpha-1}}{A_n^{\alpha}} = \frac{A_{[u]}^{\alpha}}{A_n^{\alpha}}.$$

Thus $n\chi(n) = \pi$ and

$$\int_1^n \chi(u) C_n(u)\, du = O(n^{-\alpha}) \int_1^n u^{\alpha-1}\, du = O(1) \quad \text{as } n \to \infty.$$

Hence all the hypotheses of the Theorem are satisfied and the result follows.

COROLLARY 2. (i) (SIDDIQI [**6**]). *If $\Phi(t) = o(t/\log(2\pi/t))$, as $t \to 0+$, then $\sum A_n(x)$ is summable $(N, 1/(n+1))$ to $f(x)$.*

(ii) *If $\Phi(t) = o(t/\{\log(3\pi/t)\log\log(3\pi/t)\})$, as $t \to 0+$, then $\sum A_n(x)$ is summable $(N, 1/\{(n+2)\log(n+2)\})$.*

(iii) *If $\Phi(t) = o(t/\{\log(k\pi/t)\log_2(k\pi/t)\cdots\log_q(k\pi/t)\})$, as $t \to 0+$, then $\sum A_n(x)$ is summable $(N, 1/\{(n+k)\log(n+k)\cdots\log_{q-1}(n+k)\})$, to $f(x)$, where $\log_r x = \log(\log_{r-1} x)$, for $r \geq 2$, and k is such that $\log_q k > 0$.*

PROOF. To deduce this corollary, note that, in case (i) taking

$$\chi(u) = \frac{\pi}{u \log 2u} \quad \text{and} \quad c_{n,k} = \frac{1/(n+1-k)}{\sum_0^n 1/(k+1)},$$

we obtain

$$\chi(\pi/t) = t/\log(2\pi/t),$$
$$n\chi(n) = \pi/\log 2n = o(1) \quad \text{as } n \to \infty,$$
$$C_n(u) = \sum_0^{[u]} 1/(m+1) / \sum_0^{n} 1/(k+1),$$

and thus

$$\int_1^n \chi(u) C_n(u)\, du = O\left(\frac{1}{\log n}\right) \int_1^n \frac{1}{u}\, du = O(1).$$

Thus the hypotheses of the Theorem are satisfied and the result follows.

The choice of χ, $c_{n,k}$, $C_n(u)$, etc., is similarly suggested in each of the cases (ii) and (iii), and the proof of the corollary is completed.

COROLLARY 3. *Let $\{p_n\}$ be a nonnegative, nonincreasing sequence and let $p(1/t) = p([1/t])$ and $P(1/t) = P([1/t])$.*

(i) (SINGH [**7**]). *If* (a) $\Phi(t) = o(t/\log(\pi/t))$ *as* $t \to 0+$, *and*

(b) $\sum_1^n (P_k/k \log(k+1)) = O(P_n)$,

then $\sum A_n(x)$ *is summable* (N, p) *to* $f(x)$.

(ii) (PATI [**5**]). *If* (c) $\Phi(t) = o(t/P(1/t))$ *as* $t \to 0+$, *and*

(d) $\log n = O(P_n)$,

then $\sum A_n(x)$ *is summable* (N, p) *to* $f(x)$.

(iii) (SINGH [**8**]). *If* (e) $\Phi(t) = o(p(1/t)/P(1/t))$, *as* $t \to 0+$, *then* $\sum A_n(x)$ *is summable* (N, p) *to* $f(x)$.

REMARKS. In their theorems both Pati and Singh have assumed an extra hypothesis on $\{P_n\}$: "$P_n \to \infty$, as $n \to \infty$".

PROOF. Since $\{p_n\}$ is nonnegative and nonincreasing,

$$(n+1)p_n \le p_0 + p_1 + \cdots + p_n = P_n.$$

Therefore $np_n/P_n = O(1)$, as $n \to \infty$. Taking $c_{n,k} = p_{n-k}/P_n$ we obtain

$$C_n(u) = P(u)/P_n.$$

Case (i). Take $\chi(u) = 1$, for $u \in (0, 2)$ and $\chi(u) = \pi/(u \log u)$ for $u \in [2, \infty)$. Then for $t \in (0, 1/2)$,

$$\chi(\pi/t) = t/\log(\pi/t),$$

and, for $n \ge 2$,

$$n\chi(n) = \pi/\log n.$$

Thus

$$n\chi(n) = o(1) \quad \text{as } n \to \infty.$$

Also

$$\begin{aligned}\int_1^n \chi(u)C_n(u)\,du &= \frac{P_1}{P_n} + \frac{\pi}{P_n}\int_2^n \frac{P(u)}{u\log u}\,du \\ &= \frac{P_1}{P_n} + \frac{\pi}{P_n}\sum_2^{n-1}\int_k^{k+1} \frac{P(u)}{u\,\log u}\,du \\ &\le \frac{1}{P_n}\left\{P_1 + \pi\sum_2^{n-1}\frac{P_k}{k\,\log k}\right\} \\ &\le K\left(\frac{1}{P_n}\right)\sum_1^n \frac{P_k}{k\,\log(k+1)} \\ &= O(1) \quad \text{as } n\to\infty,\end{aligned}$$

and the hypotheses of the Theorem are satisfied.

Case (ii). Take $\chi(u) = 1/uP(u)$. Then

$$n\chi(n) = 1/P(n) = O(1), \quad \text{as } n\to\infty,$$

and

$$\int_1^n \chi(u)C_n(u)\,du = \frac{1}{P_n}\int_1^n \frac{1}{u}\,du = \frac{\log n}{P_n} = O(1).$$

Case (iii). Let $\chi(u) = p(u)/P(u)$. Then

$$n\chi(n) = np_n/P_n = O(1),$$

as shown earlier, and also

$$\int_1^n \chi(u)C_n(u)\,du = \frac{1}{P_n}\int_1^n p(u)\,du = O(1).$$

Thus in each of these cases, the hypotheses of the Theorem are satisfied and the corollary follows.

COROLLARY 4 (A CONVERGENCE CRITERION). *Let χ be a decreasing function such that $\int_1^n \chi(u)\,du = O(1)$. If $\Phi(t) = o(\chi(\pi/t))$, as $t\to 0+$, then $\sum A_n(x)$ converges to $f(x)$.*

In particular, if $\chi(\pi/t)$ denotes any of the following:

(i) $t/(\log(2\pi/t))^{1+\varepsilon}$,

(ii) $t/\{\log(k\pi/t)(\log\log(k\pi/t))^{1+\varepsilon}\},\ldots$ *where $\varepsilon > 0$ and k is appropriately chosen, then $\Phi(t) = O(\chi(\pi/t))$ implies that $\sum A_n(x)$ converges to $f(x)$.*

REMARKS. This result may be compared with the corresponding classical results on nonconvergence of a Fourier series at a point of continuity, e.g. see [**10**, Theorem VIII 2.4, p. 303]. Thus, in the suggested particular cases, $\varepsilon > 0$ may not be replaced by $\varepsilon = 0$. For other alternate convergence criteria involving the case $\varepsilon = 0$, see [**3**, Theorems 3, 10; **9**, Theorems 2, 3].

We shall need the following result for a proof of Corollary 4.

LEMMA [**1**]. *Let $\{p_n\}$ satisfy the Kaluza conditions:*

$$\textit{for } n\ge 0, \quad p_n > 0 \quad \textit{and} \quad p_{n+1}/p_n \le p_{n+2}/p_{n+1} \le 1.$$

Then if $\{P_n\}$ is bounded, the method (N,p) is ineffective, i.e. only convergent sequences are summable by the method.

PROOF OF COROLLARY 4. We first note that as χ is decreasing,

$$n\chi(n) \leq \int_1^n \chi(u)\,du = O(1).$$

Now choosing $c_{n,k} = p_{n-k}/P_n$ such that $\{p_n\}$ satisfies the requirements of the Lemma (e.g. $\{p_n\}$ may be taken to be one of the sequences

$$\left\{\frac{1}{(n+1)(n+2)}\right\}, \quad \left\{\frac{1}{2^n}\right\}, \quad \left\{\frac{1}{(n+2)(\log(n+2))^{1+\varepsilon}},\ \varepsilon > 0\right\},$$

etc.), we see that the hypotheses of the Theorem are satisfied, and thus we complete the proof.

In the case of the particular instances cited, we note that

$$\Phi(t) = O(t/(\log(2\pi/t))^{1+\varepsilon}), \quad \text{as } t \to 0+$$

implies that

$$\Phi(t) = o(t/(\log(2/t))^{1+2/\varepsilon}), \quad \text{as } t \to 0+,$$

and similarly in the other cases, and then the results as claimed follow.

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