ON SUMMABILITY OF FOURIER SERIES AT A POINT

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ABSTRACT. In this paper summability of Fourier series by a regular linear method of summation determined by a triangular matrix, has been studied and various results—some known and some new—on Cesàro and Nörlund summability have been deduced. A convergence criterion has also been obtained.

1. Let $C = (c_{n,k})$, $k = 0, 1, 2, \ldots, n$, be a triangular matrix and let

$$t_n = \sum_{k=0}^{n} c_{n,k} s_k,$$

where $\{s_k\}$ is a given sequence of numbers. If $t_n \to s$ as $n \to \infty$, $\{s_n\}$ is called summable $(C)$ to $s$. In this paper we assume $c_{n,k} \geq 0$ for $k = 0, 1, 2, \ldots, n$, and $\sum_{k=0}^{n} c_{n,k} = 1$. Then a necessary and sufficient condition for regularity of the method $(C)$ is

$$\lim_{n \to \infty} c_{n,k} = 0 \quad \text{for each } k.$$

In the case

$$c_{n,k} = A^{\alpha-1}_{n-k}/A_{n}^{\alpha}, \quad \alpha \geq 0,$$

where $\{A_n^{\alpha-1}\}$ is determined by the identity

$$(1-x)^{-\alpha} = \sum_{0}^{\infty} A_n^{\alpha-1} x^n \quad (|x| < 1),$$

the method $(C)$ reduces to the well-known Cesàro method $(C, \alpha)$. For

$$c_{n,k} = p_{n-k}/P_n, \quad P_n = p_0 + p_1 + \cdots + p_n > 0,$$

the method $(C)$ reduces to the Nörlund method $(N, p)$. In the case $p_n = 1/(n+1)$, the Nörlund method $(N, 1/(n+1))$ is also known as the harmonic method.

Let $f$ be a Lebesgue integrable periodic function with period $2\pi$ and let

$$f(x) \sim \frac{1}{2} a_0 + \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{0}^{\infty} A_n(x).$$

We write

$$\phi(t) = \frac{1}{2} \{f(x + t) + f(x - t) - 2f(x)\},$$

$$\Phi(t) = \int_{0}^{t} |\phi(u)| \, du \quad \text{and} \quad s_n(x) = \sum_{0}^{n} A_k(x).$$

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Let
\[ C_n(k) = \sum_{m=0}^{k} c_{n,n-m} \]
and, for \( u \geq 0 \), define \( C_n(u) = C_n([u]) \), where \([u]\) is the greatest integer function.

Throughout the paper \( K \) is used to denote an absolute constant, not necessarily the same at each occurrence.

2. We establish the following

**THEOREM.** Let \( \{c_{n,k}\} \) be nondecreasing with respect to \( k \). Let \( \chi \) be a positive function defined over \((0, \infty)\) such that as \( n \to \infty \), (i) \( n\chi(n) = O(1) \) and (ii) \( \int_{1}^{n} \chi(u)C_n(u)\,du = O(1) \). Then if \( \Phi(t) = o(\chi(\pi/t)) \), as \( t \to 0^+ \), the series \( \sum A_n(x) \) is summable \((C)\) to \( f(x) \).

3. **Proof.** We have that \( \{c_{n,k}\} \) is nonnegative and nondecreasing in \( k \). Hence,
\[
(n-k)c_{n,k} \leq \sum_{m=k+1}^{n} c_{n,m} \leq 1.
\]
Thus for each fixed \( k \), \( c_{n,k} \to 0 \) as \( n \to \infty \), that is, \((C)\) is a regular method.

In view of the fact that the convergence of Fourier series at a point is a local property of the generating function, we may take \( \phi(t) = 0 \) over \([\delta, \pi]\), where \( 0 < \delta < \pi \). We choose \( \delta \) such that \( \Phi(t) = o(\chi(\pi/t)) \) for \( t \in (0, \delta) \). Let
\[
t_n(x) = \sum_{0}^{n} c_{n,k}s_k(x).
\]
Then we need to show that \( t_n(x) - f(x) = o(1) \) as \( n \to \infty \). After the Dirichlet integral, for \( n > \pi/\delta \),
\[
t_n(x) - f(x) = \sum_{0}^{n} c_{n,k}s_k(x) - f(x) = \frac{1}{\pi} \int_{0}^{\delta} \phi(t)L(n,t)\,dt
\]
\[= \frac{1}{\pi} \left\{ \int_{0}^{\pi/n} + \int_{\pi/n}^{\delta} \right\} = I_1 + I_2, \text{ say,}
\]
where
\[
L(n,t) = \sum_{0}^{n} \frac{c_{n,k}\sin \left(k + \frac{1}{2}\right)t}{\sin \left(\frac{1}{2}t\right)}.
\]
As
\[
|L(n,t)| \leq \pi \sum_{0}^{n} \left(k + \frac{1}{2}\right)c_{n,k} \leq \pi \left(n + \frac{1}{2}\right),
\]
we get
\[
|I_1| \leq \left(n + \frac{1}{2}\right) \int_{0}^{\pi/n} |\phi(t)|\,dt = o(n\chi(n)) = o(1),
\]
as \( n \to \infty \).

Next, in view of the order estimates of McFadden [4, Lemma 5.11],
\[
\left| \sum_{k=a}^{b} c_{n,n-k}e^{i(n-k)t} \right| \leq KC_n(\pi/t),
\]
where \( 0 \leq a \leq b \leq \infty \), \( 0 < t \leq \pi \), and \( n \) a positive integer, we obtain

\[
|I_2| \leq K \int_{\pi/n}^{\delta} \frac{\phi(t)|C_n(\pi/t)}{t} dt = K \int_{\pi/n}^{\delta} \frac{\phi(t)|C_n(\pi/t)}{t} dt + K \int_{\pi/\tau}^{\delta} \frac{\phi(t)|C_n(\pi/t)}{t} dt,
\]

where \( \tau \) is a positive integer such that \( \pi/\tau \leq \delta < \pi/\tau + 1 \). As

\[
\int_{\pi/(\tau+1)}^{\pi/\tau} \frac{\phi(t)|C_n(\pi/t)}{t} dt = \left[ \frac{C_n(\pi/t)}{t} \Phi(t) \right]_{\pi/(\tau+1)}^{\pi/\tau} + \int_{\pi/(\tau+1)}^{\pi/\tau} \frac{\Phi(t)|C_n(\pi/t)}{t^2} dt,
\]

This completes the proof of the Theorem.

4. The four corollaries in this section follow as a result of our Theorem.

**Corollary 1 (Hardy [2]).** Let \( \alpha > 0 \). If \( \Phi(t) = o(t) \), as \( t \to 0^+ \), then \( \sum A_n(x) \) is summable \((C, \alpha)\) to \( f(x) \).

The case \( \alpha = 1 \) is the classical result of Lebesgue (see [10, Theorem III 3.9]).

**Proof.** Let \( \chi(u) = \pi/u \) and \( c_{n,k} = \frac{A^{\alpha-1}_{n-k}}{A_n^\alpha} \). Then \( \chi(\pi/t) = t \) and

\[
C_n(u) = \sum_{m=0}^{[u]} c_{n,n-m} = \sum_{m=0}^{[u]} \frac{A^{\alpha-1}_m}{A_n^\alpha} = \frac{A^{\alpha}_{[u]}}{A_n^\alpha}.
\]

Thus \( n \chi(n) = \pi \) and

\[
\int_1^n \chi(u) C_n(u) du = O(n^{-\alpha}) \int_1^n u^{\alpha-1} du = O(1) \quad \text{as} \quad n \to \infty.
\]

Hence all the hypotheses of the Theorem are satisfied and the result follows.

**Corollary 2.** (i) (Siddiqi [6]). If \( \Phi(t) = o(t/\log(2\pi/t)) \), as \( t \to 0^+ \), then \( \sum A_n(x) \) is summable \((N, 1/(n + 1))\) to \( f(x) \).

(ii) If \( \Phi(t) = o(t/\{\log(3\pi/t) \log(3\pi/t)\}) \), as \( t \to 0^+ \), then \( \sum A_n(x) \) is summable \((N, 1/\{n + 2) \log(n + 2)\})\).

(iii) If \( \Phi(t) = o(t/\{\log(k\pi/t) \log^2(k\pi/t) \cdots \log_q (k\pi/t)\}) \), as \( t \to 0^+ \), then \( \sum A_n(x) \) is summable \((N, 1/\{(n + k) \log(n + k) \cdots \log_{q-1}(n + k)\})\), to \( f(x) \), where \( \log_r x = \log(\log_{r-1} x) \), for \( r \geq 2 \), and \( k \) is such that \( \log_k k > 0 \).

**Proof.** To deduce this corollary, note that, in case (i) taking

\[
\chi(u) = \frac{\pi}{u \log 2u} \quad \text{and} \quad c_{n,k} = \frac{1/(n + 1 - k)}{\sum_{r=0}^n 1/(k + 1)},
\]
we obtain
\[
\chi(\pi/t) = t/\log(2\pi/t), \\
n\chi(n) = \pi/\log 2n = o(1) \quad \text{as } n \to \infty, \\
C_n(u) = \sum_{0}^{[u]} 1/(m+1) / \sum_{0}^{n} 1/(k+1),
\]
and thus
\[
\int_{1}^{n} \chi(u)C_n(u) \, du = O\left(\frac{1}{\log n}\right) \int_{1}^{n} \frac{1}{u} \, du = O(1).
\]
Thus the hypotheses of the Theorem are satisfied and the result follows.

The choice of \(\chi, c_{n,k}, C_n(u), \) etc., is similarly suggested in each of the cases (ii) and (iii), and the proof of the corollary is completed.

**Corollary 3.** Let \(\{p_n\}\) be a nonnegative, nonincreasing sequence and let \(p(1/t) = p(\lfloor 1/t \rfloor)\) and \(P(1/t) = P(\lfloor 1/t \rfloor)\).

(i) (Singh [7]). If (a) \(\Phi(t) = o(t/\log(\pi/t)) \) as \(t \to 0^+\), and
(b) \(\sum_{0}^{n} (p_k/k \log(k+1)) = O(P_n),\)
then \(\sum A_n(x)\) is summable \((N,p)\) to \(f(x)\).

(ii) (Pati [5]). If (c) \(\Phi(t) = o(t/P(1/t)) \) as \(t \to 0^+\), and
(d) \(\log n = O(P_n),\)
then \(\sum A_n(x)\) is summable \((N,p)\) to \(f(x)\).

(iii) (Singh [8]). If (e) \(\Phi(t) = o(p(1/t)/P(1/t))\), as \(t \to 0^+\), then \(\sum A_n(x)\) is summable \((N,p)\) to \(f(x)\).

**Remarks.** In their theorems both Pati and Singh have assumed an extra hypothesis on \(\{P_n\}\): “\(P_n \to \infty\), as \(n \to \infty\).”

**Proof.** Since \(\{p_n\}\) is nonnegative and nonincreasing,
\[
(n+1)p_n \leq p_0 + p_1 + \cdots + p_n = P_n.
\]
Therefore \(np_n/P_n = O(1), \) as \(n \to \infty\). Taking \(c_{n,k} = p_{n-k}/P_n\) we obtain
\[
C_n(u) = P(u)/P_n.
\]

Case (i). Take \(\chi(u) = 1, \) for \(u \in (0, 2)\) and \(\chi(u) = \pi/(u \log u)\) for \(u \in [2, \infty)\). Then for \(t \in (0, 1/2),\)
\[
\chi(\pi/t) = t/\log(\pi/t),
\]
and, for \(n \geq 2,\)
\[
n\chi(n) = \pi/\log n.
\]
Thus
\[
n\chi(n) = o(1) \quad \text{as } n \to \infty.
\]
Also
\[ \int_1^n \chi(u) C_n(u) \, du = \frac{P_1}{P_n} + \frac{\pi}{P_n} \int_2^n \frac{P(u)}{u \log u} \, du \]
\[ = \frac{P_1}{P_n} + \frac{\pi}{P_n} \sum_{k=2}^{n-1} \frac{P_k}{k \log k} \]
\[ \leq \frac{1}{P_n} \left\{ P_1 + \pi \sum_{k=2}^{n-1} \frac{P_k}{k \log k} \right\} \]
\[ \leq K \left( \frac{1}{P_n} \right) \sum_{k=1}^{n} \frac{P_k}{k \log(k+1)} \]
\[ = O(1) \quad \text{as } n \to \infty, \]
and the hypotheses of the Theorem are satisfied.

Case (ii). Take \( \chi(u) = 1/uP(u) \). Then
\[ n\chi(n) = 1/P(n) = O(1), \quad \text{as } n \to \infty, \]
and
\[ \int_1^n \chi(u) C_n(u) \, du = \frac{1}{P_n} \int_1^n \frac{1}{u} \, du = \frac{\log n}{P_n} = O(1). \]

Case (iii). Let \( \chi(u) = p(u)/P(u) \). Then
\[ n\chi(n) = np_n/P_n = O(1), \]
as shown earlier, and also
\[ \int_1^n \chi(u) C_n(u) \, du = \frac{1}{P_n} \int_1^n p(u) \, du = O(1). \]
Thus in each of these cases, the hypotheses of the Theorem are satisfied and the corollary follows.

Corollary 4 (A Convergence Criterion). Let \( \chi \) be a decreasing function such that \( \int_1^n \chi(u) \, du = O(1) \). If \( \Phi(t) = o(\chi(\pi/t)) \), as \( t \to 0^+ \), then \( \sum A_n(x) \) converges to \( f(x) \).

In particular, if \( \chi(\pi/t) \) denotes any of the following:
(i) \( t/(\log(2\pi/t))^{1+\varepsilon} \),
(ii) \( \{\log(k\pi/t)\} \log \log(k\pi/t)^{1+\varepsilon} \), \ldots where \( \varepsilon > 0 \) and \( k \) is appropriately chosen, then \( \Phi(t) = O(\chi(\pi/t)) \) implies that \( \sum A_n(x) \) converges to \( f(x) \).

Remarks. This result may be compared with the corresponding classical results on nonconvergence of a Fourier series at a point of continuity, e.g. see [10, Theorem VIII 2.4, p. 303]. Thus, in the suggested particular cases, \( \varepsilon > 0 \) may not be replaced by \( \varepsilon = 0 \). For other alternate convergence criteria involving the case \( \varepsilon = 0 \), see [3, Theorems 3, 10; 9, Theorems 2, 3].

We shall need the following result for a proof of Corollary 4.

Lemma [1]. Let \( \{p_n\} \) satisfy the Kaluza conditions:
\[ \text{for } n \geq 0, \quad p_n > 0 \quad \text{and} \quad p_{n+1}/p_n \leq p_{n+2}/p_{n+1} \leq 1. \]
Then if \( \{P_n\} \) is bounded, the method \((N,p)\) is ineffective, i.e. only convergent sequences are summable by the method.

**Proof of Corollary 4.** We first note that as \( x \) is decreasing,

\[
nx(n) \leq \int_1^n x(u) \, du = O(1).
\]

Now choosing \( c_{n,k} = \frac{p_{n-k}}{P_n} \) such that \( \{p_n\} \) satisfies the requirements of the Lemma (e.g. \( \{p_n\} \) may be taken to be one of the sequences

\[
\left\{ \frac{1}{(n+1)(n+2)} \right\}, \quad \left\{ \frac{1}{2^n} \right\}, \quad \left\{ \frac{1}{(n+2)(\log(n+2))^{1+\epsilon}} \right\}, \quad \epsilon > 0 \}
\]

etc.), we see that the hypotheses of the Theorem are satisfied, and thus we complete the proof.

In the case of the particular instances cited, we note that

\[
\Phi(t) = O(t/(\log(2\pi/t))^{1+\epsilon}), \quad \text{as } t \to 0+
\]

implies that

\[
\Phi(t) = o(t/(\log(2/t))^{1+2/\epsilon}), \quad \text{as } t \to 0+,
\]

and similarly in the other cases, and then the results as claimed follow.

**References**


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