PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 92, Number 2, October 1984

DIEUDONNÉ-SCHWARTZ THEOREM IN INDUCTIVE LIMITS OF METRIZABLE SPACES

QIU JING-HUI

ABSTRACT. The Dieudonné-Schwartz Theorem for bounded sets in strict inductive limits does not hold for general inductive limits $E = \operatorname{ind}\lim E_n$. It does if each $\overline{E}_n^E \subset E_{m(n)}$ and all the E_n are Fréchet spaces. A counterexample shows that this condition is not necessary. When E is a strict inductive limit of metrizable spaces E_n, this condition is equivalent to the condition that each bounded set in E is contained in some E_n.

Let $E_1 \subset E_2 \subset \cdots$ be a sequence of locally convex spaces and

$$(E, \xi) = \operatorname{ind}\lim(E_n, \xi_n)$$

their inductive limit with respect to the continuous identity maps id: $(E_n, \xi_n) \to (E_{n+1}, \xi_{n+1})$. The Dieudonné-Schwartz Theorem states that a set $B \subset E$ is ξ-bounded if and only if it is contained and bounded in some (E_n, ξ_n), provided that

(H-1) each E_n is closed in (E_{n+1}, ξ_{n+1}), and

(H-2) $(E_n, \xi_n) = \xi_{n+1}|E_n$, where $\xi_{n+1}|E_n$ is the topology induced in E_n by (E_{n+1}, ξ_{n+1}).

We introduce some further hypotheses:

(H-3) each E_n is closed in (E, ξ);

(H-7) for any $n \in N$ there is $m(n) \in N$ such that $\overline{E}_n^E \subset E_{m(n)}$, where $\overline{E}_n^E$ is the closure of E_n in (E, ξ);

(H-9) for any $n \in N$ there is $m(n) \in N$ and an absolutely convex neighborhood $U^{(n)}$ of 0 in (E_n, ξ_n) such that $(\overline{U}^{(n)})^E \subset E_{m(n)}$;

(DS) each set B bounded in (E, ξ) is contained in some E_n; and

(DST) each set B bounded in (E, ξ) is contained and bounded in some (E_n, ξ_n).

The implications (H-1) and (H-2)$\Rightarrow$(H-3), (H-3)$\Rightarrow$(DS), and (H-7)$\Rightarrow$(DS) are known; see [**1**, Chapter 2, §12; **2** and **3**].

THEOREM 1. *If all* (E_n, ξ_n) *are Fréchet spaces, then* (H-7) $\Rightarrow$ (DST).

PROOF. By the hypotheses there is a sequence of natural numbers $m_1 < m_2 < \cdots$ such that $\overline{E}_1^E \subset E_{m_1}, \overline{E}_2^E \subset E_{m_2}, \ldots, \overline{E}_n^E \subset E_{m_n}, \ldots$. Since $\overline{E}_n^E$ is a closed subspace of the Fréchet space (E_{m_n}, ξ_{m_n}), $(\overline{E}_n^E, \xi_{m_n}|\overline{E}_n^E)$ is a Fréchet space. Then $(\overline{E}_1^E, \xi_{m_1}|\overline{E}_1^E) \subset (\overline{E}_2^E, \xi_{m_2}|\overline{E}_2^E) \subset \cdots$ is a sequence of Fréchet spaces, and the identity maps $(\overline{E}_n^E, \xi_{m_n}|\overline{E}_n^E) \to (\overline{E}_{n+1}^E, \xi_{m_{n+1}}|\overline{E}_{n+1}^E)$ are continuous. Since $\overline{E}_n^E$ is closed in $(\overline{E}_{n+1}^E, \xi_{m_{n+1}}|\overline{E}_{n+1}^E)$, we have $\xi_{m_n}|\overline{E}_n^E = \xi_{m_{n+1}}|\overline{E}_n^E$ by the open mapping

Received by the editors October 5, 1983.
1980 *Mathematics Subject Classification*. Primary 46A05.
Key words and phrases. Locally convex spaces, (strict) inductive limit, bounded set.

0002-9939/84 $1.00 + $.25 per page

theorem. Thus $(E,\varsigma) = \operatorname{ind}\lim(\overline{E}_n^E, \xi_{m_n}|\overline{E}_n^E)$ is the strict inductive limit and each $\overline{E}_n^E$ is closed in $(\overline{E}_{n+1}^E, \xi_{m_{n+1}}|\overline{E}_{n+1}^E)$. By the Dieudonné-Schwartz Theorem, each bounded set in (E,ς) is contained and bounded in some $(\overline{E}_n^E, \xi_{m_n}|\overline{E}_n^E)$. Evidently, $E_n \subset \overline{E}_n^E$ and $\xi_n \supset \xi_{m_n}|E_n$, so $(E,\xi) \supset (E,\varsigma)$; see [**4**, Chapter V, Supplement (3)]. Let $B \subset E$ be ξ-bounded; then B is ς-bounded. Thus B is contained and bounded in some $(\overline{E}_n^E, \xi_{m_n}|\overline{E}_n^E)$ and B is contained and bounded in (E_{m_n}, ξ_{m_n}).

COUNTEREXAMPLE. Let $R_+ = [0,+\infty)$, $E_n = \{f$ is a real-valued measurable function on R_+: $\int_0^{+\infty} e^{-2nx}[f(x)]^2\,dx < +\infty\}$. All (E_n,ξ_n) are Hilbert spaces with the inner product $(f,g) \mapsto \int_0^{+\infty} e^{-2nx} f(x)g(x)\,dx$, $E_1 \subset E_2 \subset \cdots$, and id: $(E_n,\xi_n) \to (E_{n+1},\xi_{n+1})$ are continuous. By [**5**, Theorem 4], (DST) holds. We show that (H-7) does not hold. Assume that $\overline{E}_n^E \subset E_m$ for some $m > n$. Take a, b such that $b < n < m < a < m+1$. Then $e^{ax} \in E_{m+1} \setminus E_m$. The functions

$$f_k(x) = \begin{cases} e^{ax}, & 0 \le x \le k, \\ e^{bx}, & k < x, \end{cases}$$

all belong to E_n and converge in (E_{m+1},ξ_{m+1}) to e^{ax}. Then f_k converges in (E,ξ) to e^{ax}, so $e^{ax} \in \overline{E}_n^E \subset E_m$. This contradicts $e^{ax} \notin E_m$.

The counterexample shows that hypothesis (H-7) in Theorem 1 is not a necessary condition for (DST). However, for strict inductive limits of metrizable spaces, we have (H-7)$\Leftrightarrow$(DS).

LEMMA 1. *Let* $(E,\xi) = \operatorname{ind}\lim(E_n,\xi_n)$ *be a strict inductive limit of metrizable spaces. Then* (H-7) $\Leftrightarrow$ (H-9).

PROOF. It is obvious that (H-7)$\Rightarrow$(H-9). Assume that $U^{(n)}$ is an absolutely convex neighborhood of 0 in (E_n,ξ_n) and $(\overline{U}^{(n)})^E \subset E_{m(n)}$. There is an absolutely convex open neighborhood U of 0 in (E,ξ) such that $U^{(n)} \supset (U \cap E_n)$. Then $(\overline{U \cap E_n})^E \subset (\overline{U}^{(n)})^E \subset E_{m(n)}$. For any $x \in U \cap \overline{E}_n^E$, there is a net $\{x_\delta\} \subset E_n$ such that $x_\delta \to x$ in (E,ξ). Since U is a neighborhood of x in (E,ξ) there is δ_0 such that $x_\delta \in U$ for any $\delta \ge \delta_0$. Then $x_\delta \in U \cap E_n$ for any $\delta \ge \delta_0$ and $x \in (\overline{U \cap E_n})^E$. From this, $U \cap \overline{E}_n^E \subset (\overline{U \cap E_n})^E \subset E_{m(n)}$ and $\overline{E}_n^E = \bigcup_{k=1}^\infty K(U \cap \overline{E}_n^E) \subset E_{m(n)}$. Namely, (H-9)$\Rightarrow$(H-7).

THEOREM 2. *Let* $(E,\xi) = \operatorname{ind}\lim(E_n,\xi_n)$ *be a strict inductive limit of metrizable spaces. Then* (H-7) $\Leftrightarrow$ (DS).

PROOF. From [**3**], (H-7)$\Rightarrow$(DS). We need only prove (DS)$\Rightarrow$(H-7). Assume (H-7) does not hold, i.e. there is some $n \in N$ such that $\overline{E}_n^E$ is not contained in any E_m. By Lemma 1, for each absolutely convex neighborhood $U^{(n)}$ of 0 in (E_n,ξ_n), $(\overline{U}^{(n)})^E$ is not contained in any E_m. Let d_n be a linear metric defining the topology ξ_n. Put $U_k^{(n)} = \{x \in E_n\colon d_n(0,x) < 1/k\}$. Then $(\overline{U}_k^{(n)})^E$ is not contained in any E_m for each $K \in N$. Thus there is $x_1 \in (\overline{U}_1^{(n)})^E \setminus E_1$. Suppose $x_1 \in E_{m_2}$, there is $x_2 \in (\overline{U}_2^{(n)})^E \setminus E_{m_2}$. Suppose $x_2 \in E_{m_3}$, there is $x_3 \in (\overline{U}_3^{(n)})^E \setminus E_{m_3}, \ldots$. We repeat this process, obtaining a sequence $\{x_k\colon k \in N\} \subset E$ such that $x_k \in (\overline{U}_k^{(n)})^E \setminus E_{m_k}$; here $m_1 = 1 < m_2 < m_3 < \cdots$. Let U be any absolutely convex neighborhood of 0 in (E, ξ); there is $x'_k \in U_k^{(n)}$ such that $x_k \in x'_k + U$ for each $k \in N$. Since $d_n(0, x'_k) \xrightarrow{k} 0$, $\{x'_k\colon k \in N\}$ is bounded in (E_n, ξ_n) and bounded in (E, ξ). There is $\lambda > 0$ such that $\{x'_k\colon k \in N\} \subset \lambda U$. Therefore

$$\{x_k\colon k \in N\} \subset \{x'_k\colon k \in N\} + U \subset \lambda U + U = (\lambda + 1)U.$$

That is, $\{x_k\colon k \in N\}$ is bounded in (E, ξ) and not contained in any E_m.

ACKNOWLEDGEMENT. The author is grateful to the referee for many valuable suggestions.

REFERENCES

1. J. Horváth, *Topological vector spaces and distributions*, Vol. 1, Addision-Wesley, Reading, Mass., 1966.
2. J. Kucera and K. McKennon, *Bounded sets in inductive limits*, Proc. Amer. Math. Soc. **69** (1978), 62–64.
3. J. Kucera and C. Bosch, *Dieudonné-Schwartz theorem on bounded sets in inductive limits*. II, Proc. Amer. Math. Soc. **86** (1982), 392–394.
4. A. P. Robertson and W. J. Robertson, *Topological vector spaces*, Cambridge Univ. Press, 1964.
5. J. Kucera and K. McKennon, *Dieudonné-Schwartz theorem on bounded sets in inductive limits*, Proc. Amer. Math. Soc. **78** (1980), 366–368.

BEIJING UNIVERSITY OF IRON & STEEL TECHNOLOGY, ROOM 404, BUILDING NO. 10, BEIJING, PEOPLE'S REPUBLIC OF CHINA