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L_p SMOOTHNESS AND APPROXIMATE CONTINUITY

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ABSTRACT. It is known that a measurable smooth function can have only countably many points of discontinuity. A measurable function is constructed here having the property of being L_p smooth and having uncountably many points of L_p (in fact, approximate) discontinuity.

A real-valued function f defined on the real line R is said to be *smooth at the point* $x \in R$ if

$$\Delta^2 f(x,t) = o(t) \quad \text{as } t \to 0,$$

where $\Delta^2 f(x,t) = f(x+t) + f(x-t) - 2f(x)$. Similarly, a measurable function f is said to be L_p $(1 \leq p < \infty)$ smooth at the point $x \in R$ if

$$\left\{ \frac{1}{t} \int_0^t |\Delta^2 f(x,h)|^p \, dh \right\}^{1/p} = o(t) \quad \text{as } t \to 0.$$

We call f *smooth* (*L_p smooth*) if it is smooth (L_p smooth) at each $x \in R$. The continuity properties of measurable smooth and L_p smooth functions have been studied extensively. Neugebauer [**3**, **4**] showed that if f is a measurable smooth function, then $R \backslash C(f)$ is a nowhere dense countable set, where $C(f)$ denotes the set of points at which f is continuous. Evans and Larson [**2**] have recently shown that $R \backslash C(f)$ can be characterized by a *clairsemé* (scattered) set. O'Malley [**5**] has shown that an L_p smooth function f must belong to class Baire* one and, hence, $R \backslash C(f)$ is nowhere dense; however, Neugebauer [**4**] has shown that $R \backslash C(f)$ can have large measure. Nonetheless, for an L_p smooth function f, $R \backslash L_pC(f)$ must be of measure zero, where $L_pC(f)$ denotes the set of points at which f is L_p continuous; i.e., those points x at which

$$\left\{ \frac{1}{t} \int_0^t |f(x+h) - f(x)|^p \, dh \right\}^{1/p} = o(1) \quad \text{as } t \to 0.$$

As Neugebauer [**4**] mentions, a natural question is whether the nowhere dense and measure zero set $R \backslash L_pC(f)$ must be countable. The purpose of the present paper is to show that it need not.

In particular, for any $1 \leq p < \infty$ we shall construct a function f which is L_p smooth, but for which $R \backslash \mathrm{AC}(f)$ is uncountable, where $\mathrm{AC}(f)$ denotes the set of points at which f is approximately continuous; i.e., those points x at which for each $\varepsilon > 0$ the set $\{h \colon |f(x+h) - f(x)| \geq \varepsilon\}$ has 0 at a point of dispersion. It is easily seen that $L_pC(f) \subseteq \mathrm{AC}(f)$.

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In some sense the function constructed in this paper is an improvement on the function exhibited in [**1**]. That function has the properties of being approximately smooth and possessing uncountably many points of approximate discontinuity.

In what follows we shall let $m(E)$ denote the Lebesgue measure of a measurable set E. We shall assume that p is a natural number in the construction of the example. Naturally, one may obtain an L_q smooth example for an arbitrary $q \geq 1$ simply by applying the following result with p being the smallest natural number greater than or equal to q.

THEOREM. *For each $1 \leq p < \infty$ there is an L_p smooth function which is approximately discontinuous at uncountably many points.*

PROOF. Let p be a natural number. We shall first describe a certain symmetric Cantor set C in $[0, 1]$. Our function will be approximately discontinuous at each point of C.

For each nonnegative integer m, let

$$L(m) = \prod_{k=0}^{m} (k+1)^{-(2p+2)^{2(m-k+1)}}.$$

The set C is defined as the intersection of a nested sequence of compact sets, I_n, where each I_n is the union of 2^n closed subintervals of I_{n-1}, each subinterval is of length $L(n)$, and each shares an endpoint with one of the intervals comprising I_{n-1}; more specifically, if $N(m) = L(m-1)/L(m)$, then

$$I_n = \left\{ \sum_{m=1}^{\infty} k(m)L(m) \colon k(m) = 0 \text{ or } N(m) - 1 \text{ for } 1 \leq m \leq n , \right.$$
$$\left. \text{and } 0 \leq k(m) < N(m) \text{ for } m > n \right\},$$

and

$$C = \left\{ \sum_{m=1}^{\infty} k(m)L(m) \colon k(m) = 0 \text{ or } N(m) - 1 \right\}.$$

For any specific nonnegative integer n, $R \backslash I_n$ consists of $2^n + 1$ open intervals, whose union we denote by CI_n. We will define the function f in an inductive manner using the sets CI_n; the function f restricted to C will be identically zero. However, at the nth stage of the construction, f will not be defined on all of CI_n, but only on a certain relatively large (and increasingly larger with n) open subset of CI_n. This subset, denoted by J_n, is defined as follows.

Let I be the union of a finite collection of disjoint open intervals, say $I = \bigcup_{k \in P}(a_k, b_k)$, where P is a finite set of natural numbers. If $0 < r < (b_k - a_k)/2$ for all $k \in P$, then we set

$$I^r = \bigcup_{k \in P} (a_k + r, b_k - r).$$

Using this notation, we define

$$J_n = (CI_n)^{M(n)L(n)},$$

where $M(n) = (L(n))^{(-2p+1)/(2p+2)}$.

We are now in a position to define the function f and we begin by defining it on $\bigcup_{n=0}^{\infty} J_n = R\backslash C$.

First, to define f on J_0, we note that J_0 consists of the two intervals $(-\infty, -1)$ and $(2, +\infty)$. We define f to be -1 on the former at 1 on the latter.

Next, we describe the inductive step of the definition of f. Suppose that f has been defined on J_n. Now, $R\backslash J_n$ consists of 2^n intervals, each centered on an interval of I_n, and each of length $(2M(n)+1)L(n)$. Denote one such interval by I and partition I into $2M(n)+1$ intervals of equal length. Code the middle open interval with the number 1 and then code each of the remaining $2M(n)$ open intervals with a -1 or a 1 so that the signs alternate. For $x \in J_{n+1}\backslash J_n$, let $f(x)$ be the value of the code assigned to the interval to which x belongs. Now, this procedure fails to assign a value to $f(x)$ at finitely many points $x \in J_{n+1}\backslash J_n$. At each such point, define $f(x)$ in such a way to make f continuous at x if that is possible, and define $f(x) = 0$ otherwise. Let A_{n+1} denote those points of $J_{n+1}\backslash J_n$ where we have assigned f the value zero.

In this manner f is defined at every point of $\bigcup_{n=0}^{\infty} J_n = R\backslash C$, and we complete the definition by defining f to be 0 on C.

We now verify that f has the desired properties. First, notice that if $x \in R\backslash(C \cup (\bigcup_{n=1}^{\infty} A_n))$ then there is a neighorhood of x on which f is a constant function, of value either -1 or 1. Consequently, f is both smooth and continuous at every such point. Furthermore, if $x \in \bigcup_{n=1}^{\infty} A_n$, then there are intervals (a, x) and (x, b) such that f has absolute value one on $(a,b)\backslash\{x\}$ and opposite signs on (a, x) and (x, b). Hence f is smooth at x. Next, since f is zero at each point of C, but $|f|$ is one almost everywhere, we clearly have that f fails to be approximately continuous at each point of C. It remains to show that f is L_p smooth at each point of C.

To this end, for each $x \in R$ and $t > 0$ we set

$$\psi_x(t) = \left\{ \frac{1}{t^{p+1}} \int_0^t |f(x+h) + f(x-h)|^p \, dh \right\}^{1/p},$$

and propose to show that, for $x \in C$,

$$\lim_{t \to 0} \psi_x(t) = 0;$$

i.e., that f is L_p smooth at x. We shall accomplish this by showing that for $x \in I_n$ and $t \in [L(n-1), L(n-2)]$, $\psi_x(t) < 64/n^2$ for each $n \geq 3$.

Fix an $n \geq 3$ and let $x \in I_n$. From the symmetry of the construction we may assume that $x \in [0, L(n)]$. Set

$$V_1 = [(M(n)+2)L(n), L(n-1) - (M(n)+1)L(n)],$$
$$V_2 = [L(n-1) + M(n)L(n), 2L(n-1)],$$

and for $3 \leq k \leq M(n-1)$ set

$$V_k = [(k-1)L(n-1) + 2L(n), kL(n-1)].$$

Also set

$$\mathcal{V} = [(M(n-1)+1)L(n-1), L(n-2) - (M(n-1)+1)L(n-1)].$$

Let $T_i = V_i - x$, $i = 1, 2, \ldots, M(n-1)$, and $\mathcal{T} = \mathcal{V} - x$. For $h \in T_i$ we have $f(x+h) = (-1)^{i-1}$ and $f(x-h) = (-1)^i$. Furthermore, if $h \in \mathcal{T}$, then $f(x+h) = 1$ and $f(x-h) = -1$. Consequently, if we set

$$\nu_x(t) = 2\left[\frac{m([0,t]\backslash((\bigcup_{i=1}^{M(n-1)} T_i) \cup \mathcal{T}))}{t^{p+1}}\right]^{1/p},$$

then for $t \in [L(n-1),\ L(n-2)]$ we have $\psi_x(t) \le \nu_x(t)$.

Hence, it will suffice to show that $\nu_x(t) \le 64/n^2$ for $t \in [L(n-1), L(n-2)]$. It is clear that for $t \in [L(n-1), L(n-2)]$,

$$\nu_x(t) \le \max\{\nu_x(t_1),\ \nu_x(t_2), \nu_x(t_3)\},$$

where $t_1 = L(n-1) + M(n)L(n) - x$, $t_2 = (M(n-1)+1)L(n-1) - x$ and $t_3 = L(n-2) - x$. It will, therefore, be enough to show that ν_x is bounded above by $64/n^2$ at each of these three critical numbers.

In the computations which follow, we shall utilize the following, easily verifiable equality:

$$\frac{L(n)}{(L(n-1))^{(2p+2)^2}} = \frac{1}{(n+1)^{(2p+2)^2}}.$$

First,

$$\begin{aligned}
\nu_x(t_1) &= 2\left[\frac{m([0,t_1]\backslash T_1)}{t_1^{p+1}}\right]^{1/p} \\
&= 2\left[\frac{t_1 - (L(n-1) - (2M(n)+3)L(n))}{t_1^{p+1}}\right]^{1/p} \\
&\le 2\left[\frac{L(n-1) + M(n)L(n) - (L(n-1) - (2M(n)+3)L(n))}{t_1^{p+1}}\right]^{1/p} \\
&= 2\left[\frac{3L(n)(M(n)+1)}{t_1^{p+1}}\right]^{1/p} < 2\left[\frac{6L(n)M(n)}{(L(n-1))^{p+1}}\right]^{1/p} \\
&= 2\left[\frac{6(L(n))^{1/(2p+2)}}{(L(n-1))^{p+1}}\right]^{1/p} = 2\left[\frac{6L(n)}{(L(n-1))^{2(p+1)^2}}\right]^{1/2p(p+1)} \\
&\le 12\left[\frac{L(n)}{(L(n-1))^{(2p+2)^2}}\right]^{1/2p(p+1)} = 12\left[\frac{1}{(n+1)^{(2p+2)^2}}\right]^{1/2p(p+1)} \\
&< \frac{12}{(n+1)^2}.
\end{aligned}$$

Next,

$$\begin{aligned}
\nu_x(t_2) &= 2\left[\frac{m([0,t_2])\setminus\bigcup_{i=1}^{M(n-1)} T_i}{t_2^{p+1}}\right]^{1/p}\\
&= 2\left[\frac{t_2-[M(n-1)L(n-1)-(3M(n)+2M(n-1)-1)L(n)]}{t_2^{p+1}}\right]^{1/p}\\
&\le 2\left[\frac{L(n-1)+(3M(n)+2M(n-1)-1)L(n)}{t_2^{p+1}}\right]^{1/p}\\
&< 2\left[\frac{L(n-1)+5M(n)L(n)}{t_2^{p+1}}\right]^{1/p}\\
&< 2\left[\frac{2^{p+1}L(n-1)}{(M(n-1)L(n-1))^{p+1}}\right]^{1/p}+2\left[\frac{2^{p+1}5M(n)L(n)}{(M(n-1)L(n-1))^{p+1}}\right]^{1/p}\\
&\le 8[L(n-1)^{1/2}]^{1/p}+40\left[\frac{L(n)^{1/(2p+2)}}{(L(n-1))^{(p+1)/(2p+2)}}\right]^{1/p}\\
&< \frac{8}{n^2}+\frac{40}{(n+1)^2}<\frac{48}{n^2}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\nu_x(t_3) &= 2\left[\frac{m([0,t_3])\setminus((\bigcup_{i=1}^{M(n-1)} T_i)\cup \mathcal{T})}{t_3^{p+1}}\right]^{1/p}\\
&= 2\left[\frac{t_3-[L(n-2)-(M(n-1)+2)L(n-1)-(3M(n)+2M(n-1)-1)L(n)]}{t_3^{p+1}}\right]^{1/p}\\
&< 2\left[\frac{L(n-1)+(3M(n)+2M(n-1)-1)L(n)}{t_2^{p+1}}\right]^{1/p}+2\left[\frac{(M(n-1)+1)L(n-1)}{t_3^{p+1}}\right]^{1/p}\\
&< \frac{48}{n^2}+2\left[\frac{2^{p+2}M(n-1)L(n-1)}{(L(n-2))^{p+1}}\right]^{1/p}\le\frac{48}{n^2}+16\left[\frac{(L(n-1))^{1/(2p+2)}}{(L(n-2))^{p+1}}\right]^{1/p}<\frac{64}{n^2}.
\end{aligned}$$

This completes the proof of the Theorem.

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