SINGULAR FUNCTIONS AND DIVISION IN $H^\infty + C$

PAMELA B. GORKIN

Abstract. In this paper it is shown that for each inner function $u$, there exists a singular inner function $S$ which is divisible in $H^\infty + C$ by all positive powers of $u$.

Introduction. In this paper, we continue the study of division in $H^\infty + C$ begun by Guillory and Sarason. We let $H^\infty$ denote the space of boundary functions for bounded analytic functions in the open unit disk $D$ and $C$ denote the space of continuous, complex valued functions on $\partial D$. We let $L^\infty$ denote the usual Lebesgue space with respect to Lebesgue measure. It is well known that $H^\infty + C$ is a closed subalgebra of $L^\infty$. The space $H^\infty$ (or $H^\infty + C$) will be identified with its analytic (or harmonic) extension to $D$.

C. Guillory and D. Sarason began the study of division in $H^\infty + C$ by determining a criterion for deciding whether an $H^\infty + C$ function is divisible by all positive powers of a unimodular $H^\infty + C$ function [3]. In the same paper, the question of finding, for each inner function $u$, a singular inner function which is divisible in $H^\infty + C$ by all positive powers of $u$, is posed. We shall answer this question affirmatively. The techniques used to prove this are a combination of the techniques used in [1 and 3]. As in [1], our main tools are interpolating Blaschke products and the Chang-Marshall Theorem. A sequence $\{z_n\}$ of distinct points in $D$ is called an interpolating sequence if there exists $\delta > 0$ such that

\[
\prod_{j \neq k} \left| \frac{z_k - z_j}{1 - \overline{z_j}z_k} \right| \geq \delta > 0, \quad k = 1, 2, 3, \ldots.
\]

It is well known [4, p. 199] that if a sequence of points $\{z_n\}$ of the open unit disk is an interpolating sequence, then

\[\sum_{k=1}^{\infty} \left( 1 - |z_k|^2 \right) < \infty.\]

A Blaschke product with a zero sequence which is an interpolating sequence is called an interpolating Blaschke product.

The Chang-Marshall Theorem states that every closed subalgebra of $L^\infty$ which contains $H^\infty$ is generated by $H^\infty$ and some collection of conjugates of interpolating Blaschke products.
From the proof of the Chang-Marshall Theorem, it is easy to show that the closed subalgebra generated by $H^\infty$ and the conjugate of one inner function is actually equal to the closed algebra generated by $H^\infty$ and the conjugate of a single interpolating Blaschke product. We refer the reader to [2, Chapter IX].

**The Main Theorem.** In this section we prove the following theorem:

**Main Theorem.** For each inner function $u$, there exists a singular inner function which is divisible in $H^\infty + C$ by all positive powers of $u$.

The proof of the Main Theorem requires three lemmas. Lemmas 1 and 2 below reduce the problem to the case in which $u$ is an interpolating Blaschke product. We then use Lemma 3 to complete the proof of the Main Theorem.

**Lemma 1.** Let $u$ be an inner function. There exists an interpolating Blaschke product $b$ such that if an inner function $v$ is divisible in $H^\infty + C$ by all positive powers of $b$, then $v$ is divisible in $H^\infty + C$ by all positive powers of $u$.

**Proof.** It follows from (the proof of) the Chang-Marshall Theorem that there exists an interpolating Blaschke product $b$ such that the closed subalgebra of $L^\infty$ generated by $H^\infty$ and $\bar{u}$ is actually equal to the closed subalgebra generated by $H^\infty$ and the conjugate of the interpolating Blaschke product $b$. Let $v$ be an inner function divisible by all positive powers of $b$. It is easy to see that $v$ must be divisible in $H^\infty + C$ by all positive powers of $u$.

The maximal ideal space of $H^\infty$, denoted $M(H^\infty)$, is the set of nonzero complex multiplicative linear functionals on $H^\infty$. With the weak-* topology, $M(H^\infty)$ is a compact Hausdorff space. We identify $D$ with its natural image in $M(H^\infty)$.

**Lemma 2.** Let $b$ be an interpolating Blaschke product with zero sequence $\{z_n\}$. If $S$ is a singular inner function such that $S(z_n) \to 0$, then $S$ is divisible by all positive powers of $b$.

**Proof.** For each positive integer $n$, let $g_n$ be an analytic $n$th root of $S$. Thus $g_n^* = S$, $g_n \in H^\infty$ and, for each $n$, $g_n(z_m) \to 0$ as $m \to \infty$. Suppose $m \in M(H^\infty) \sim D$ and $m(b) = 0$. By [4, p. 205], we have $m \in \{z_n\}$. Hence $m(g_n) = 0$. It follows from Lemma 1 of [1] that $g_n(b) \in H^\infty + C$. Thus $g_n^* b^n \in H^\infty + C$ for each $n$ and $Sb^n \in H^\infty + C$, as desired.

The techniques used to construct the singular function $S$ are similar to those used in [3]. The construction will be done on the upper half-plane.

**Lemma 3.** Let $\{z_n\}$ be an interpolating Blaschke sequence. There exists a singular inner function $S$ satisfying $S(z_n) \to 0$.

**Proof.** If $A = \{n: \Re z_n \geq 0\}$ is finite, then we need only consider the set $\{z_n\}$ such that $\Re z_n < 0$. Assume there are infinitely many $z_n$ such that $\Re z_n \geq 0$. For those $n$, let $w_n = i((1 - z_n)/(1 + z_n))$. Then $\Im w_n > 0$ and, from (*), we have $\sum_n \Im w_n < \infty$. Let $\{b_n\}$ be a sequence of positive real numbers such that
\[ \sum_n b_n'(\text{Im } w_n) < \infty \text{ and } \lim_{n \to \infty} b_n = \infty. \] Let \( w'_n = \text{Re } w_n + ib_n \text{Im } w_n \) and \( t_n = \text{Re } w_n. \) Finally, let \( u \) be the Poisson integral of the measure \( \mu = \sum_n (\text{Im } w_n') \delta_{t_n}, \) that is
\[ u(x, y) = \int_{-\infty}^{\infty} \frac{y}{(x-t)^2 + y^2} \, d\mu(t) = y \sum_n \frac{b_n'(\text{Im } w_n)}{(x-t_n)^2 + y^2}. \]
Then
\[ \int_{-\infty}^{\infty} \frac{d\mu(t)}{1 + t^2} = \sum_n \frac{b_n \text{Im } w_n}{1 + (t_n)^2} \]
and since \( \sum_n b_n \text{Im } w_n/(1 + t_n^2) \leqslant \sum_n b_n \text{Im } w_n \) we have \( \sum_n b_n \text{Im } w_n/(1 + t_n^2) \to \infty. \)

(b) \[ u(\text{Re } w_m, \text{Im } w_m) = \sum_n \frac{b_n'(\text{Im } w_n)(\text{Im } w_m)}{(\text{Re } w_m - t_n)^2 + (\text{Im } w_m)^2} \]
so we have
\[ u(\text{Re } w_m, \text{Im } w_m) > \frac{b_m'(\text{Im } w_m)^2}{(\text{Re } w_m - \text{Re } w_m)^2 + (\text{Im } w_m)^2} = b_m. \]
Let \( \tilde{u} \) be the harmonic conjugate of \( u, \) and let \( S_1 = e^{-(u + i\tilde{u})} \) denote the singular inner function for the upper half-plane corresponding to \( \mu. \) Then \( |S_1(w_m)| = |e^{-u(w_m)}| < e^{-b_m/2}. \) Hence \( S_1(w_m) \to 0 \) as \( m \to \infty. \) Letting \( S_2(z) = S_1((i - z)/(i + z)) \) we obtain a singular inner function such that \( S_2(z_n) \to 0 \) as \( n \to \infty \) and \( n \in A. \)

Suppose now that \( \{ n: \text{Re } z_n < 0 \} \) is infinite. Let \( w_n = i((1 + z_n)/(1 - z_n)) \) for all \( n \) such that \( \text{Re } z_n < 0. \) Again, \( \text{Im } w_n > 0 \) and \( \sum_n \text{Im } w_n < \infty. \) Repeating the process above, we obtain a singular inner function \( S_3 \) such that \( z_n \) with \( \text{Re } z_n < 0 \) we have \( S_3(z_n) \to 0 \) as \( n \to \infty. \) If we let \( S = S_2S_3, \) then \( S \) satisfies the desired conditions.

To establish the Main Theorem, let \( u \) be an inner function. Choose an interpolating Blaschke product \( b \) satisfying the conditions of Lemma 1. Use Lemma 3 to obtain a singular function \( S \) satisfying the conditions of Lemma 2. Then \( b^nS \in H^\infty + C \) for all positive integers \( n. \) By Lemma 1 we see that \( S \) is divisible by all positive powers of \( u. \)

**References**


Department of Mathematics, Bucknell University, Lewisburg, Pennsylvania 17837