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M-ACCRETIVE OPERATORS WITH M-DISPERSIVE RESOLVENTS[1]

RALPH DELAUBENFELS

ABSTRACT. We characterize linear m-accretive operators with m-dispersive resolvents. T is linear and m-accretive, with $(\lambda + T)^{-1}$ m-dispersive, if and only if the sequence $\left\langle n!\phi(\lambda + T)^{n+1}x\right\rangle_{n=0}^{\infty}$ equals the moments of a positive measure on the positive real line, for sufficiently many ϕ in X^*, x in X.

Introduction. Classical analysis is often very useful in linear operator theory. In another paper, we show that T is a spectral operator of scalar type when the sequence $\langle \phi(T^n x)\rangle_{n=0}^{\infty}$ equals the moments of a positive measure, for sufficiently many ϕ in X^*, x in X. In this paper, we consider the sequence $\left\langle n!\phi(\lambda + T)^{n+1}x\right\rangle_{n=0}^{\infty}$, for some positive λ, and again apply the classical analysis of the moment problem.

We say that a sequence of real numbers $\langle a_n\rangle_{n=0}^{\infty}$ satisfies a *Stieltjes moment condition* if $\Sigma\alpha_k\overline{\alpha_j}a_{k+j} \geqslant 0$ and $\Sigma\alpha_k\overline{\alpha_j}a_{k+j+1} \geqslant 0$, for all finite sequences $\langle \alpha_k\rangle$ of complex numbers. These conditions are satisfied if and only if there exists a positive measure μ so that

$$a_n = \int_0^{\infty} t^n \, d\mu(t) \quad \text{for all } n \qquad (\text{see } [\mathbf{3}]).$$

We will say $\langle a_n\rangle_{n=0}^{\infty}$ is *Stieltjes* if, in addition, $\Sigma_n \inf_{k \leqslant n}|a_k|^{-1/2k}$ is infinite. When $\langle a_n\rangle_{n=0}^{\infty}$ is Stieltjes, the positive measure μ is unique.

If T is a linear operator, then a vector x is *quasi-analytic for* T if $\Sigma_n \inf_{k \leqslant n}\|T^k x\|^{-1/k}$ is infinite. $\mathscr{D}_q(T)$ is the set of all quasi-analytic vectors for T (see [**1**]).

T is *m-accretive* if it is densely defined and $(-T)$ generates a 1-parameter contraction semigroup. If the semigroup is positive, on a Banach lattice, T is *m-dispersive*.

See [**2**] for other terminology in operator theory, such as accretive and core.

THEOREM. *Suppose X is a Banach lattice, T is closed, densely defined and accretive, and there exists $\mathscr{D} \subseteq X^+ \cap \mathscr{D}_q(T)$ such that $(\lambda + T)(\mathscr{D}) \subseteq \mathscr{D}$ and span $(\mathscr{D})$ is a core for $(\lambda + T)^2$, for some $\lambda > 0$. Then T is m-accretive, with $(\lambda + T)^{-1}$ m-dispersive if and only if the sequence $\left\langle n!\phi(\lambda + T)^{n+1}x\right\rangle_{n=0}^{\infty}$ satisfies a Stieltjes moment condition for all ϕ in $(X^*)^+$, x in $\mathscr{D}$.*

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PROOF. Suppose T is m-accretive, with $(\lambda + T)^{-1}$ m-dispersive.

We have, for all ϕ in X^*, x in the domain of T, the resolvent formula

$$n!\phi(\lambda + T)^{n+1}x = \int_0^\infty s^n \phi(e^{-s(\lambda+T)^{-1}}x)\, ds.$$

When $\phi \in (X^*)^+$, $x \in \mathscr{D}$, then $e^{-s(\lambda+T)^{-1}}$, the semigroup generated by $(\lambda + T)^{-1}$, is positive, so $\phi(e^{-s(\lambda+T)^{-1}}x)\, ds$ is a positive measure. Thus, $\left\langle n!\phi(\lambda + T)^{n+1}x\right\rangle_{n=0}^\infty$ satisfies a Stieltjes moment condition. (Note that the density of the domain of T was not needed here.)

Conversely, suppose $\left\langle n!\phi(\lambda + T)^{n+1}x\right\rangle_{n=0}^\infty$ satisfies a Stieltjes moment condition, for all ϕ in $(X^*)^+$, x in $\mathscr{D}$.

Since $x \in \mathscr{D}_q(T)$, the sequence is Stieltjes. Thus, if $\phi \in (X^*)^+$, $x \in \mathscr{D}$, there exists a unique positive measure $\mu_{\phi,x}$ such that

$$(1) \qquad n!\phi(\lambda + T)^{n+1}x = \int_0^\infty t^n\, d\mu_{\phi,x}(t) \quad \text{for all } n.$$

Note that

$$\int_0^\infty t^n \int_t^\infty d\mu_{\phi,x}(s)\,dt = \int_0^\infty \int_0^s t^n\, dt\, d\mu_{\phi,x}(s)$$
$$= \int_0^\infty \frac{s^{n+1}}{n+1}\, d\mu_{\phi,x}(s) = n!\phi(\lambda + T)^{n+2}x$$
$$= \int_0^\infty t^n\, d\mu_{\phi,(\lambda+T)x}(t).$$

Since the positive measure in (1) is unique, we have

$$(2) \qquad \int_t^\infty d\mu_{\phi,x}(s)\, dt = d\mu_{\phi,(\lambda+T)x}(t), \quad \text{for all } \phi \text{ in } (X^*)^+,\ x \in \mathscr{D}.$$

This implies that $d\mu_{\phi,(\lambda+T)^2x}(t)/dt$ exists and is a continuous function of t. For each t, define $F(t)\colon (\lambda + T)^2(\mathscr{D}) \to X^{**}$ by

$$(3) \qquad [F(t)y](\phi) \equiv \frac{d\mu_{\phi,y}(t)}{dt},$$

if ϕ is positive. For arbitrary $\phi \in X^*$, there exist positive ϕ_1, ϕ_2, ϕ_3, ϕ_4, with ϕ_1 orthogonal to ϕ_2, ϕ_3 orthogonal to ϕ_4 and $\phi = (\phi_1 - \phi_2) + i(\phi_3 - \phi_4)$. Then

$$[F(t)y]\phi \equiv [F(t)y]\phi_1 - [F(t)y]\phi_2 + i([F(t)y]\phi_3 - [F(t)y]\phi_4).$$

To see that $F(t)y \in X^{**}$, note that if $\phi \in (X^*)^+$,

$$|F(t)(\lambda + T)x(\phi)| = \int_t^\infty d\mu_{\phi,x}(s) \leqslant \int_0^\infty d\mu_{\phi,x}(s) = \phi(\lambda + T)x.$$

Since $F(t)(\lambda + T)x$ is positive, this implies that $\|F(t)(\lambda + T)x\| \leqslant \|(\lambda + T)x\|$. Thus $\|F(t)\| \leqslant 1$, for all $t \geqslant 0$.

Note that $F(0)(\lambda + T)x = \int_0^\infty F(s)x\, ds = (\lambda + T)x$. Thus $F(0) = I$. Collecting all the facts about $F(t)$: For all $y \in (\lambda + T)^2(\mathscr{D})$, $\phi \in (X^*)^+$,

$$(4) \qquad [F(t)y](\phi) \text{ is positive continuous function of } t,$$

$$(5) \qquad F(0) = I,$$

(6) $$n!\phi(\lambda + T)^{n+1}y = \int_0^\infty t^n[F(t)y](\phi)\,dt,$$

(7) $$\int_t^\infty [F(s)y](\phi)\,ds = [F(t)(\lambda + T)y](\phi).$$

To show that T is m-accretive, we need to show that the range of $(\lambda + T)$ is dense. First, we will show that the range of $(\lambda + T)^3$ is dense in the range of $(\lambda + T)^2$, written $R(\lambda + T)^2$.

So suppose $\phi \in [R(\lambda + T)^3]^\perp$, the annihilator of $R(\lambda + T)^3$. There exist positive $\phi_1, \phi_2, \phi_3, \phi_4$ with ϕ_1 orthogonal to ϕ_2, ϕ_3 orthogonal to ϕ_4, so that $\phi = (\phi_1 - \phi_2) + i(\phi_3 - \phi_4)$.

Suppose $y \in (\lambda + T)^2(\mathscr{D})$. Then, for all n, $0 = n!\phi(\lambda + T)^{n+1}y$. By (6), this implies that

$$\int_0^\infty t^n[F(t)y](\phi_1)\,dt = \int_0^\infty t^n[F(t)y](\phi_2)\,dt.$$

Since $[F(t)y](\phi_i)$ is positive, for $i = 1, 2$, the uniqueness of the positive measure in (1) implies that $[F(t)y]\phi_1\,dt = [F(t)y]\phi_2\,dt$. Since $F(t)$ is continuous, this implies that $[F(t)y]\phi_1 = [F(t)y]\phi_2$; in particular,

$$\phi_1(y) = [F(0)y]\phi_1 = [F(0)y]\phi_2 = \phi_2(y).$$

By an identical argument, $\phi_3(y) = \phi_4(y)$. Thus $\phi(y) = 0$.

Since the span of $\mathscr{D}$ is a core for $(\lambda + T)^2$, $(\lambda + T)^2(\mathscr{D})$ is total in $R(\lambda + T)^2$. Since ϕ annihilates the total set $(\lambda + T)^2(\mathscr{D})$, $\phi \in [R(\lambda + T)^2]^\perp$.

Thus, since T is closed, $R(\lambda + T)^2 = R(\lambda + T)^3$.

Since T is accretive, this implies that T is m-accretive, on $R(\lambda + T)^2$. This implies that $(\lambda + T)^{-1}$ is a bounded operator on $R(\lambda + T)^2$. Thus, if x is in the domain of T, then $x = (\lambda + T)^{-2}(\lambda + T)^2 x$ is in $R(\lambda + T)^2$. Since the domain of T is dense, and T is closed, $R(\lambda + T)^2$, and therefore $R(\lambda + T)$, equals X. Thus T is m-accretive.

To show that T has an m-dispersive resolvent, we will extend $F(t)$ to all of $\mathscr{D}$, and show that it equals the semigroup generated by $(\lambda + T)^{-1}$.

Differentiating both sides of (7) gives

(8) $$-F(t)y = F'(t)(\lambda + T)y \quad \text{for all } y \in (\lambda + T)^2(\mathscr{D}).$$

This implies that $F(t)y = e^{-t(\lambda+T)^{-1}}y$ (the semigroup exists, because $(\lambda + T)^{-1}$ is bounded). Thus $F(t)(\lambda + T)^2 x$ is a C^∞ function of t, for all x in $\mathscr{D}$.

By (2), this implies that $d\mu_{\phi,x}(t)/dt$ exists, and is a C^∞ function of t, for all x in $\mathscr{D}$, ϕ in $(X^*)^+$.

Define $G(t)$: $\mathscr{D} \to X^{**}$ exactly as $F(t)$ was defined in (3). Equations (4)–(8) all hold for $G(t)$.

Since $e^{-t(\lambda+T)^{-1}}x$ (equal to $G(t)x$) is positive, for all x in $\mathscr{D}$, and $\bar{\mathscr{D}}$ equals X^+, $e^{-t(\lambda+T)^{-1}}$ is positive, so that $(\lambda + T)^{-1}$ is m-dispersive, concluding the theorem.

The hypotheses are clearly satisfied when T is bounded and accretive (letting $\mathscr{D} = X^+$). Note that T is automatically m-accretive.

COROLLARY. *Suppose X is a Banach lattice and T is bounded and accretive. Then $(\lambda + T)^{-1}$ is m-dispersive if and only if $\left\langle n!\phi(\lambda + T)^{n+1}x\right\rangle_{n=0}^{\infty}$ satisfies a Stieltjes moment condition for all ϕ in $(X^*)^+$, x in X^+.*

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DEPARTMENT OF MATHEMATICS, DARTMOUTH COLLEGE, HANOVER, NEW HAMPSHIRE 03755

Current address: Department of Mathematical Sciences, The University of Tulsa, 600 South College Avenue, Tulsa, Oklahoma 74104