A CHAOTIC FUNCTION WHOSE NONWANDERING SET IS THE CANTOR TERNARY SET

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Abstract. We introduce a continuous function from [0,1] onto itself whose non-wandering set is the Cantor ternary set C. This function is also chaotic in the sense of Li and Yorke, but with the property that almost all points of [0,1] are eventually fixed. Those points of C which are periodic, eventually periodic, asymptotically periodic or recurrent are also explicitly identified.

Let \( f: [0,1] \rightarrow [0,1] \) be defined by \( f(x) = -3x + 1 \) if \( 0 < x < 1/3 \), \( f(x) = 0 \) if \( 1/3 \leq x \leq 2/3 \), and \( f(x) = 3x - 2 \) if \( 2/3 < x \leq 1 \). Then \( x = 1/13 \) is a periodic point of \( f(x) \) with minimal period 3. Continuous functions from [0,1] into itself with a periodic point whose period is not an integral power of 2 have been called chaotic by several people [1, 3–5]. So this function \( f(x) \) is chaotic in that sense. In the sequel, \( f(x) \) is always the above-quoted function.

Recall that the Cantor ternary set C is obtained by first removing the (open) middle third \((1/3, 2/3)\) from \([0,1]\), then removing the (open) middle thirds \((1/9, 2/9)\) and \((7/9, 8/9)\) of the remaining 2 intervals, and so on. For our function \( f(x) \), it is easy to see that \( f(x) \) maps the (closed) middle third \([1/3, 2/3]\) of \([0,1]\) onto the (unstable) fixed point \( x = 1 \) and \( f^2(x) \) maps the (closed) middle thirds \([1/9, 2/9]\) and \([7/9, 8/9]\) of the remaining 2 intervals (in addition to the first closed middle third \([1/3, 2/3]\)) onto the fixed point \( x = 1 \), and so on. Since the Cantor set C has (Lebesgue) measure zero, almost all points of \([0,1]\) are mapped onto the same and unstable fixed point \( x = 1 \) by \( f(x) \) after only finitely many iterations. Therefore, from a physical point of view, this function \( f(x) \) is not chaotic after all (see [2] for more of this). This seems to suggest that chaotic functions should be further classified (see [4] also).

From the above argument, we also obtain that the Cantor ternary set C is invariant under \( f(x) \) and the nonwandering set \( \Omega \) of \( f(x) \) is a subset of C. In this note, we identify those points of \( \Omega \) which are periodic, eventually periodic, asymptotically periodic or recurrent. As a consequence, we obtain that \( P = R = \Omega = C \), where \( P \) (\( R \), respectively) is the set of all periodic (recurrent, respectively) points of \( f(x) \).

For every real number \( x \) in the Cantor ternary set C, there is a unique ternary expansion. That is, \( x = \sum_{n=1}^{\infty} (a_n)/(3^n) \), where \( a_n = 0 \) or 2 for all \( n \geq 1 \). We shall write \( x = a_1 a_2 a_3 \cdots \) from now on. In the sequel, we only consider those points of
[0,1] which are in C. So when we write $x = b_1b_2b_3 \cdots$, we always mean that $b_i = 0$ or 2 for all $i \geq 1$, and $x$ is the real number whose (unique) ternary expansion is $b_1b_2b_3 \cdots$. Let $A = a_1a_2a_3 \cdots$. If the sequence $\{a_n\}$ is periodic, i.e., $a_i = a_{m+i}$ for some $m \geq 1$ and all $i \geq 1$, then we write $A = a_1 \cdots a_m$. The notation $B = b_1b_2b_3 \cdots = c_1c_2 \cdots c_n A$ will mean that $b_i = c_i$ for all $i = 1, 2, \ldots, n$, and $b_{n+k} = a_k$ for all $k \geq 1$. So $2A$ is the real number whose ternary expansion is $2a_1a_2a_3 \cdots$. It does not mean 2 times $A$. For $A = a_1a_2a_3 \cdots$, we let $A' = a_1' a_2' a_3' \cdots$ denote the real number whose ternary expansion is $(2 - a_1)(2 - a_2)(2 - a_3) \cdots$. That is, $a_i' = 2 - a_i$ for all $i \geq 1$. With the above notation, we have $f(2A) = A$ and $f(0A) = A'$.

Now we can state the following theorem. The proof is easy and omitted.

**Theorem.** Let $A = a_1a_2a_3 \cdots$. Then the following hold.

(a) $A$ is periodic with period $n$ (need not be minimal) if and only if $A = a_1 \cdots a_n$ with $a_n = 2$, or $A = a_1 \cdots a_n a'_1 \cdots a'_n$ with $a_n = 0$.

(b) $A$ is eventually periodic with period $n$ (need not be minimal) if and only if $A = a_1 \cdots a_m b_1 \cdots b_n$ or $A = a_1 \cdots a_m b'_1 \cdots b'_n b_1 \cdots b_n$ for some integer $m \geq 1$ and $b_i = 0$ or 2 for all $i = 1, 2, \ldots, n$, with $b_n = a_m$.

(c) $A$ is asymptotically periodic with period $n$ (need not be minimal) if and only if $A = a_1 \cdots a_m b_1 \cdots b_n$ or $A = a_1 \cdots a_m b'_1 \cdots b'_n b_1 \cdots b_n$ for some integer $m \geq 1$ and $b_i = 0$ or 2 for all $i = 1, 2, \ldots, n$, with $b_n = a_m$.

(d) $A$ is recurrent if and only if, for every (large) integer $n > 1$, there exists an integer $m(n) > 1$ such that at least one of the following holds.

(1) If $a_{m(n)} = 2$, then $a_{m(n)+1} = a_i$ for all $i = 1, 2, \ldots, n$.

(2) If $a_{m(n)} = 0$, then $a_{m(n)+1} = a'_i$ for all $i = 1, 2, \ldots, n$.

(e) $\tilde{P} = \tilde{R} = \tilde{\Omega} = C$, where $P$, $R$, $\Omega$ and $C$ are defined as before.

**References**