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A STRONG LAW OF LARGE NUMBERS FOR MARTINGALES[1]

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ABSTRACT. We derive a moment inequality for the Skorohod representation theorem and apply it to obtain a strong law of large numbers for martingales.

1. Introduction. In the early sixties, Skorohod proved the following result; see Skorohod [**5**, p. 163], Hall and Heyde [**4**, p. 269].

THEOREM 1. *Let $\{S_n = \sum_{j=1}^n X_j, n \geq 1\}$ be a zero mean martingale. There exists a probability space supporting a standard Brownian motion $\{B(t)\}$ and a sequence of nonnegative random variables $\{\tau_n, n \geq 1\}$ such that*

(i) *$\{B(\sum_{j=1}^n \tau_j)\}$ has the same joint distribution as $\{S_n\}$,*
(ii) *$E(\tau_n^r) \leq C_r E(|X_n|^{2r})$, C_r is a constant depending on r, $r \geq 1$.*

In this paper, we shall derive an inequality more general than Theorem 1(ii) and apply this inequality to prove a strong law of large numbers for martingales.

2. A moment inequality. We shall adopt the following representation method which is slightly different from the one given by Freedman [**3**, p. 68]. Let X be a zero mean random variable with distribution function F, let $f(x) = F^{-1}(x) = \inf\{t: F(t) \geq x\}$, $0 \leq x \leq 1$. Let $g(x)$ be a monotone solution of the function equation $\int_x^{g(x)} f(t)\,dt = 0$. Now construct a probability space supporting a standard Brownian motion process $\{B(t)\}$ and a random variable U which is independent of $\{B(t)\}$ and uniformly distributed over $[0,1]$. Conditioning on $U = x$, define $\tau(x) = \inf\{t: B(t) = f(x) \text{ or } f(g(x))\}$. Then the distribution of B_τ is F and $E(\tau) = E(X^2)$. To see this, let $G(u,v)$ be the distribution which assign probability $|u|/(|u|+|v|)$ at v and probability $|v|/(|u|+|v|)$ at u, $uv < 0$. Note that conditioning on $U = x$, B_τ has distribution $G(f(x), f(g(x)))$. Therefore, if ψ is a bounded measurable function, then

$$E\psi(B_\tau) = \int_0^1 [\psi(f(x))|f(g(x))| + \psi(f(g(x)))|f(x)|](|f(g(x))| + |f(x)|)^{-1}\,dx.$$

Using the facts that

$$g(g(x)) = x, \qquad f(x) = f(g(x))g'(x)$$

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we see

$$\int_0^1 \psi(f(g(x)))|f(x)|(|f(x)|+|f(g(x))|)^{-1}\,dx$$
$$= \int_0^1 \psi(f(x))|f(x)|(|f(x)|+|f(g(x))|)^{-1}\,dx.$$

Hence

$$E\psi(B_\tau) = \int_0^1 \psi(f(x))\,dx = \int_{-\infty}^{\infty} \psi(x)\,dF(x).$$

That is, B_τ has distribution F. By the same reason,

$$E(\tau) = \int_0^1 |f(x)f(g(x))|\,dx = \int_0^1 f^2(x)\,dx = \int_{-\infty}^{\infty} x^2\,dF(x) = E(X^2).$$

Moreover, we shall show

THEOREM 2. *Let $\phi(x)$, $x \geq 0$, be a nonnegative function such that for some a, b, $\frac{1}{2} < a < b < \infty$, $\phi(x)/x^a$ is nondecreasing and $\phi(x)/x^b$ is nonincreasing. Then $E\phi(\tau) \leq CE\phi(X^2)$, where the constant C depends only on a and b.*

PROOF. Following Breiman [**1**], let $T = \inf\{t: B(t) = u \text{ or } v\}$, $-u \leq 0 \leq v$. Then

$$P(T \geq t) = \frac{4}{\pi}\sum_{n=1}^{\infty}(2n+1)^{-1}\exp\left[-\frac{\lambda_n^2 t}{2r^2}\right]\sin\left(\frac{\lambda_n u}{r}\right),$$

where $\lambda_n = (2n+1)$, $r = u+v$. By induction, $\sin(\lambda_n u/r) \leq 4\pi^2(2n+1)uv/r^2$. Hence,

$$E\phi(T) = \int_0^\infty P(T \geq t)\,d\phi(t)$$
$$\leq C_1 \frac{uv}{r^2}\int_0^\infty \left(\sum_{n=1}^{\infty}\exp\left[-\frac{\lambda_n^2 t}{2r^2}\right]\right)\,d\phi(t).$$

From the inequality

$$\sum_{n=0}^{\infty} e^{-\lambda_n^2 s} \leq e^{-8s}\sum_{n=0}^{\infty} e^{-4n^2\pi^2 s}$$
$$\leq e^{-8s}\left(1+\int_0^\infty e^{-4\pi^2 s x^2}\,dx\right) = \left(1+\frac{C_2}{\sqrt{s}}\right)e^{-8s}$$

follows

$$E\phi(T) \leq C_3\left[\frac{uv}{r^2}\int_0^\infty \exp\left[-\frac{4t}{r^2}\right]\,d\phi(t) + \frac{uv}{r}\int_0^\infty \exp\left[-\frac{4t}{r^2}\right]\frac{d\phi(t)}{\sqrt{t}}\right].$$

Write

$$\int_0^\infty e^{-st}\,d\phi(t) = \int_0^{1/s} e^{-st}\,d\phi(t) + \int_{1/s}^\infty e^{-st}\,d\phi(t),$$

where

$$\int_0^{1/s} e^{-st}\,d\phi(t) = e^{-1}\phi\left(\frac{1}{s}\right) + s\int_0^{1/s} e^{-st}\phi(t)\,dt$$
$$\leq e^{-1}\phi(s^{-1}) + s^{a+1}\phi(s^{-1})\int_0^{1/s} t^a e^{-st}\,dt$$
$$\leq C_4\phi(s^{-1})$$

and

$$\int_{1/s}^{\infty} e^{-st}\,d\phi(t) = -e^{-1}\phi(s^{-1}) + s\int_{1/s}^{\infty} e^{-st}\phi(t)\,dt$$
$$\le -e^{-1}\phi(s^{-1}) + s^{b+1}\phi(s^{-1})\int_{1/s}^{\infty} t^b e^{-st}\,dt$$
$$\le C_5\phi(s^{-1}).$$

Similarly

$$\int_0^{\infty} t^{-1/2}e^{-st}\,d\phi(t) = \int_0^{1/s} t^{-1/2}e^{-st}\,d\phi(t) + \int_{1/s}^{\infty} t^{-1/2}e^{-st}\,d\phi(t)$$
$$\le C_6\sqrt{s}\phi(s^{-1}),$$

where C_1, C_2, C_3, C_4, C_5, C_6 are constants and C_4, C_5 depend on a, C_6 depends on b. Therefore, one concludes

$$E\phi(T) \le C\frac{uv}{r^2}\phi\left(\frac{r^2}{4}\right) \le C\frac{uv}{r^2}\phi\left(\frac{u^2+v^2}{2}\right),$$

where C is a constant depending on a and b.

Note

$$\phi\left(\tfrac{1}{2}(u^2+v^2)\right) \le \max\{\phi(u^2),\phi(v^2)\} \le \phi(u^2)+\phi(v^2).$$

Hence,

$$E\phi(T) \le C\frac{uv}{r^2}(\phi(u^2)+\phi(v^2)).$$

This implies

$$E\phi(\tau) \le C\int_0^1 \frac{|f(x)f(g(x))|}{(|f(x)|+|f(g(x))|)^2}[\phi(f^2(x))+\phi(f^2(g(x)))]\,dx$$
$$= C\int_0^1 |f(x)|\phi(f^2(x))(|f(x)|+|f(g(x))|)^{-1}\,dx$$
$$\le C\int_0^1 \phi(f^2(x))\,dx = CE\phi(X^2). \quad \text{Q.E.D.}$$

3. A strong law. Using the result in Theorem 2, we shall prove a strong law of large numbers for martingales. First, observe a simple fact: if two sequences of random variables $\{X_n\}$ and $\{Y_n\}$ have the joint distribution, C is a constant, then $P(|X_n - C| \ge \varepsilon, \text{i.o.}) = P(|Y_n - C| \ge \varepsilon, \text{i.o.})\ \forall\varepsilon > 0$; therefore, $X_n \to C$, a.s. if and only if $Y_n \to C$, a.s.

THEOREM 3. *Let $S_n = \{\sum_{j=1}^n X_j, n \ge 1\}$ be a zero mean martingale. Let $\phi(x)$, $x \ge 0$, satisfy the assumptions in Theorem 2 and $\phi(x+y) \le \phi(x)+\phi(y)$, for all x, $y \ge 0$. If for some sequence of positive numbers $\{a_n\}$, we have $a_n \uparrow \infty$ and $\sum_{n=1}^{\infty} E\phi(X_n^2)/\phi(a_n) > \infty$, then $S_n/a_n^p \to 0$ a.s. for every $p > \frac{1}{2}$.*

PROOF. Let $\{B(\sum_{j=1}^n \tau_j)\}$ represent $\{S_n\}$ as in Theorem 1. By Theorem 2 and the assumption that $\sum_{n=1}^{\infty} E\phi(X_n^2)/\phi(a_n) < \infty$, we conclude

$$\sum_{n=1}^{\infty} \phi(\tau_n)/\phi(a_n) < \infty \quad \text{a.s.}$$

Applying Kronecker's Lemma we get

$$\frac{1}{\phi(a_n)}\left(\sum_{j=1}^{n}\phi(\tau_j)\right)\to 0 \quad \text{a.s.}$$

By the assumption $\phi(x+y)\le\phi(x)+\phi(y)$, we see

$$\phi\left(\sum_{j=1}^{n}\tau_j\right)/\phi(a_n)\to 0 \quad \text{a.s.}$$

Because $\phi(x)/x^a$ is nondecreasing and $\phi(x)/x^b$ is nonincreasing, we have

$$\phi\left(\sum_{j=1}^{n}\tau_j\right)/\phi(a_n)\ge \min\left\{\left[\frac{1}{a_n}\left(\sum_{j=1}^{n}\tau_j\right)\right]^a,\left[\frac{1}{a_n}\left(\sum_{j=1}^{n}\tau_j\right)\right]^b\right\}.$$

Therefore,

$$\frac{1}{a_n}\left(\sum_{j=1}^{n}\tau_j\right)\to 0 \quad \text{a.s.}$$

Without loss of generality, we may assume $S_n\to\infty$ a.s. which implies $\sum_{j=1}^{n}\tau_j\to\infty$ a.s. But the law of the iterated logarithm for Brownian paths tells us

$$B\left(\sum_{j=1}^{n}\tau_j\right)\Bigg/\left(\sum_{j=1}^{n}\tau_j\right)^p\to 0 \quad \text{a.s. } \forall p>\frac{1}{2}.$$

Hence,

$$S_n/a_n^p\to 0 \quad \text{a.s. } \forall p>\tfrac{1}{2}. \quad \text{Q.E.D.}$$

COROLLARY 4. *Let $\{S_n=\sum_{j=1}^{n}X_j\}$ be a zero mean martingale. If $E(X_n^2)<\infty$ $\forall n$, and $s_n^2=\sum_{j=1}^{n}E(X_j^2)\to\infty$, then $S_n/s_n^p\to 0$, a.s. $\forall p>1$.*

PROOF. Choose $\phi(x)=x$, $a_n=s_n^2(\log s_n^2)^{1+\varepsilon}\uparrow\infty$, $\varepsilon>0$. Then

$$\sum_{n=1}^{\infty}\frac{E\phi(X_n^2)}{\phi(a_n)}=\sum_{n=1}^{\infty}\frac{E(X_n^2)}{s_n^2(\log s_n^2)^{1+\varepsilon}}<\infty$$

by Dini's theorem, and consequently

$$S_n/s_n^{2p}(\log s_n^2)^{p+p\varepsilon}\to 0 \quad \text{a.s. for every } p>\tfrac{1}{2}$$

which implies

$$S_n/s_n^p\to 0 \quad \text{a.s. for every } p>1. \quad \text{Q.E.D.}$$

COROLLARY 5. *Let $\{S_n=\sum_{j=1}^{n}X_j\}$ be a zero mean martingale. If*

$$\sum_{n=1}^{\infty}E|X_n|^r/n^r<\infty, \qquad 1<r\le 2,$$

then $S_n/n^p\to 0$ a.s. $\forall p>1$.

PROOF. Choose $\phi(x) = x^{r/2}$, $a_n = n^2 \uparrow \infty$. Apparently, $\phi(x)$ satisfies the required conditions in Theorem 3 and

$$\sum_{n=1}^{\infty} \frac{E\phi(X_n^2)}{\phi(a_n)} = \sum_{n=1}^{\infty} E|X_n|^r/n^r < \infty$$

by assumption. Hence $S_n/n^p \to 0$ a.s. for every $p > 1$. Q.E.D.

REMARK 6. Chow [**2**] has shown that if $\sum_{n=1}^{\infty} E(X_n^2)/n^2 < \infty$, then $S_n/n \to 0$ a.s.

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