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ADDING A 2-HANDLE TO A 3-MANIFOLD: AN APPLICATION TO PROPERTY R

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ABSTRACT. A sufficient condition is given for adding a 2-handle to a 3-manifold and obtaining a 3-manifold with incompressible boundary. The main result is then used to show that a knot in the 3-sphere admitting a partially unknotted spanning surface has Property R.

In [**P**] it was shown that if J is a simple closed curve in the boundary of a 3-dimensional handlebody H and $\partial H - J$ is incompressible in H, then the 3-manifold obtained by adding a 2-handle to H along J has incompressible boundary. The proof in [**P**] was algebraic. It used the Freiheitssatz from one-relator group theory and was predicated on the fact that the fundamental group of a 3-dimensional handlebody is a finitely generated free group.

In this note the above result is generalized to a larger class of 3-manifolds, which includes 3-dimensional handlebodies. The proof is geometric. The main theorem is applied to give a new class of knots that have Property R.

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1. Main theorem. Let M be a 3-manifold, J a two-sided simple closed curve (s.c.c.) in ∂M and B a 3-cell with the product structure $B = \mathbf{D}^2 \times I$. The *3-manifold obtained by adding a 2-handle to M along J* is the 3-manifold that is the identification space obtained from the disjoint union of M and B via an identification of the annulus $\partial \mathbf{D}^2 \times I \subset \partial B$ with an annular neighborhood of J in ∂M. A simple closed curve K in $\partial M - J$ is *coplanar* (in ∂M) *with* J if K separates $\partial M - J$ into two components, one of which is planar. If M is compact and K is coplanar with J, then precisely one of the following possibilities occurs: K bounds a disk in $\partial M - J$, K is parallel in ∂M to J, or K separates ∂M into two components where one is a once-punctured torus T and $J \subset T$ does not separate T. A properly embedded planar surface P in M is a *pre-disk with respect to J* if:

(1) $\partial P \subset \partial M - J$,

(2) one component, s, of ∂P is *not* coplanar with J, and

(3) each compoent of $\partial P - s$ is coplanar with J.

LEMMA 1. *Let M be a 3-manifold and let J be a simple closed curve in ∂M such that $\partial M - J$ is incompressible. If M has compressible boundary, then there is no pre-disk in M with respect to J.*

PROOF. Consider the collection $\mathcal{P}$ of properly embedded, connected, planar surfaces in M where $P \in \mathcal{P}$ if there is a s.c.c. $J \subset \partial M$, $\partial M - J$ is incompressible

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and P is a pre-disk with respect to J. The conclusion of the lemma is that the collection $\mathcal{P}$ is empty. So, suppose that $\mathcal{P}$ is not empty.

Let $P \in \mathcal{P}$ be chosen so that $\chi(P)$ is a maximum of the values of the Euler characteristics of the members of $\mathcal{P}$. There is a s.c.c. $J \subset \partial M$, $\partial M - J$ is incompressible and P is a pre-disk with respect to J.

Let $\mathcal{D}$ be the collection of disks in M so that $D \in \mathcal{D}$ if D is a properly embedded disk in M and ∂D is *not* contractible in ∂M. By hypothesis the collection $\mathcal{D}$ is not empty. Since $\partial M - J$ is incompressible, any disk in $\mathcal{D}$ must meet J. Furthermore, if K is a s.c.c. in $\partial M - J$, K is coplanar with J and K is not contractible in M, then $\partial M - K$ is incompressible; and so, any disk in $\mathcal{D}$ must meet K. It follows that any disk in $\mathcal{D}$ must meet P.

There is no loss to assume that each disk in $\mathcal{D}$ is transverse to P. Choose a disk $D \in \mathcal{D}$ so that the number of components of $D \cap P$ is a minimum for the number of components of $D' \cap P$ where D' ranges over the members of $\mathcal{D}$.

The proof is to analyze every possible component of the intersection $D \cap P$. Each leads to a contradiction to the choice of either P or D; hence, ultimately each leads to a contradiction of the existence of P.

One component of ∂P is a s.c.c. in $\partial M - J$ that is *not* coplanar with J. This special curve will be denoted by s throughout the argument. Each assertion that follows presumes the preceding assertions.

(1) ASSERTION. *No component of $D \cap P$ is a simple closed curve.*

If this were not the case, then there would be a component α of $D \cap P$ that is a s.c.c. and α is "innermost on D"; i.e., α bounds a disk Δ in D and $\mathring{\Delta} \cap P = \varnothing$. The curve α divides P into the two planar surfaces P_1 and P_2 with $s \in P_1$.

If P_2 is not a disk, then $P' = P_1 \cup \Delta$ is a pre-disk with respect to J and $\chi(P') > \chi(P)$. This contradicts the choice of P.

If P_2 is a disk, then there is a component β of $D \cap P$ that is a s.c.c. and β is "innermost on P_2"; i.e. β bounds a disk Δ' in P_2 and $\mathring{\Delta}' \cap D = \varnothing$. The curve β divides D into an annulus D_1 and a disk D_2 with $\partial D \subset D_1$. The disk $D' = D_1 \cup \Delta'$ has the property that (after a small isotopy) the number of components of $D' \cap P$ is smaller than the number of components of $D \cap P$. This contradicts the choice of D.

(2) ASSERTION. *No component of $D \cap P$ is an inessential spanning arc on P.*

If this were not the case then there would be a component α of $D \cap P$ that is a spanning arc of both D and P and an arc β in ∂P such that $\partial \alpha = \alpha\beta$ and the curve $\alpha \cup \beta$ bounds a disk Δ in P with $\mathring{\Delta} \cap D = \varnothing$. A boundary compression of D at α using the disk Δ results in two properly embedded disks D_1 and D_2 in M with the property that the number of components of $D_i \cap P$ is smaller than the number of components of $D \cap P$, $i = 1, 2$. Since ∂D is not contractible in ∂M, at least one of the disks D_i has noncontractible boundary in ∂M. This contradicts the choice of D.

Before continuing the analysis of the components of $D \cap P$, there is an important observation to make. Namely, *it can be assumed that some component of ∂P is parallel in ∂M to J.*

First, recall that P is not a disk, each component of $\partial P - s$ is coplanar with J, and no component of ∂P is contractible in ∂M. So, if it were the case that no component of ∂P is parallel in ∂M to J, then there would be a s.c.c. $K \subset \partial M$, where K separates ∂M into two components with one a once-punctured torus T,

$J \subset T$ does not separate T, and, moreover, each component of $\partial P - s$ is parallel in ∂M to K. The planar surface P is a pre-disk with respect to K and $\partial M - K$ is incompressible. So, we just replace J by K and proceed with the analysis.

For the remainder of the argument let α be a component of $D \cap P$ that is "outermost on D"; i.e. there is an arc $\beta \subset \partial D$ such that $\partial\alpha = \partial\beta$ and the curve $\alpha \cup \beta$ bounds a disk Δ in D with $\mathring{\Delta} \cap P = \varnothing$. The important point here is that by assuming that some component of ∂P is parallel to J in ∂M, it follows that the arc $\beta \subset \partial D$ *does not meet* J.

(3) Assertion. *The component α of $D \cap P$ is not a spanning arc with both end points in s.*

If this were the case, then a boundary compression of P at α using the disk Δ results in the two new planar surfaces P_1 and P_2 where $\chi(P_i) > \chi(P)$ for $i = 1, 2$. Furthermore, at least one of the planar surfaces P_i is a pre-disk with respect to J. This contradicts the choice of P.

(4) Assertion. *The component α of $D \cap P$ is not a spanning arc with one end point in s and the other end point in a component $\mathfrak{p}$ of $\partial P - s$.*

If this were the case, then a boundary compression of P at α using the disk Δ results in a new planar surface P' where $\chi(P') > \chi(P)$. The boundary of P' consists of each component of $\partial P - \{s \cup \mathfrak{p}\}$ (each of which is coplanar with J) and a new s.c.c. s', where s' is homotopic in ∂M to the composition $s\beta\mathfrak{p}\beta^{-1}$, which is *not* coplanar with J. Hence P' is a pre-disk with respect to J. This contradicts the choice of P.

(5) Assertion. *The component α of $D \cap P$ is not a spanning arc with one end point in a component $\mathfrak{p}$ of $\partial P - s$ and the other end point in a component $\mathfrak{p}'$ of $\partial P - s$, where $\mathfrak{p}' \neq \mathfrak{p}$.*

If this were the case, then a boundary compression of P at α using the disk Δ results in a new planar surface P' where $\chi(P') > \chi(P)$. The boundary of P' contains s, each component of $\partial P - \{s \cup \mathfrak{p} \cup \mathfrak{p}'\}$ (each of which is coplanar with J) and a new s.c.c. $\mathfrak{p}''$, where $\mathfrak{p}''$ is homotopic in ∂M to the composition $\mathfrak{p}\beta\mathfrak{p}'\beta^{-1}$, which *is coplanar* with J. Hence, P' is a pre-disk with respect to J. This contradicts the choice of P.

(6) Assertion. *The component α of $D \cap P$ is not a spanning arc with both end points in a component $\mathfrak{p}$ of $\partial P - s$.*

If this were the case, then a boundary compression of P at α using the disk Δ results in two new planar surfaces P_1 and P_2, where $s \subset \partial P_1$. Furthermore, $\chi(P_i) > \chi(P)$, $i = 1, 2$.

Now, the end points of α separate $\mathfrak{p}$ into two arcs $\mathfrak{p}_1$ and $\mathfrak{p}_2$ where $\mathfrak{p}_i \subset P_i$ $(i = 1, 2)$. The boundary of P_1 contains s, some components of $\partial P - \{s \cup \mathfrak{p}\}$ (each of which is coplanar with J) and a new component s_1, which is a s.c.c. homotopic in ∂M to the composition $\mathfrak{p}_1\beta$. The boundary of P_2 contains some components of $\partial P - \{s \cup \mathfrak{p}\}$ (each of which is coplanar with J) and a new component s_2, which is a s.c.c. homotopic in ∂M to the composition $\mathfrak{p}_2\beta$.

If s_2 is *not* coplanar with J, then P_2 is a pre-disk with respect to J. This contradicts the choice of P. So, assume that s_2 is coplanar with J.

Now, if s_2 is coplanar with J, then s_1 must also be coplanar with J (since $\mathfrak{p} = p_1p_2^{-1}$ is coplanar with J). It follows that P_1 is a pre-disk with respect to J. This also contradicts the choice of P.

This completes the analysis. Each case leads to a contradiction; so, there is no pre-disk with respect to J. $\square$

THEOREM 2. *Let M be a 3-manifold and let J be a simple closed curve in ∂M such that $\partial M - J$ is incompressible. If M has compressible boundary, then the 3-manifold obtained by adding a 2-handle to M along J has incompressible boundary.*

PROOF. A boundary contraction in the 3-manifold obtained by adding a 2-handle to M along J implies the existence of a pre-disk in M with respect to J. This contradicts Lemma 1. $\square$

2. Applications to Property R. A knot k in S^3 had *Property R* if 0-frame surgery on k gives a manifold distinct from $S^1 \times S^2$. It is conjectured that if k is not the trivial knot in S^3, then k has Property R (see [**K-M**, **L**, **M**]). A surface $S \subset S^3$ with boundary the knot k is a *spanning surface for k*. A spanning surface S for k is *incompressible* if $S - k$ is incompressible in $S^3 - k$. A spanning surface S for k is *unknotted* if the fundamental group of $S^3 - S$ is a free group and is *partially unknotted* if the fundamental group of $S^3 - S$ is a free product.

THEOREM 3. *If the knot k in S^3 has an orientable, incompressible, partially unknotted spanning surface, then k has Property R.*

PROOF. Let $U(k)$ be an open tubular neighborhood of k in S^3 and let $N(k) = S^3 - U(k)$. If S is an orientable, incompressible spanning surface for k, then $S(k) = S - U(k)$ is a two-sided, incompressible surface in $N(k)$ with boundary the unique longitude L of $N(k)$. The manifold $\hat{M}$ obtained by 0-frame surgery on k is the manifold obtained by adding a 2-handle to $N(k)$ along L and adding a 3-cell (3-handle) to close up the resulting 2-sphere boundary. Furthermore, there is an obvious closed surface $\hat{S}$ in $\hat{M}$ obtained by adding a disk (2-handle) to the boundary of $S(k)$ in $\hat{M}$.

Now, what actually follows from Theorem 2 is that $\hat{S}$ is an incompressible surface in $\hat{M}$. Hence, $\hat{M}$ cannot be $S^1 \times S^2$.

To see this, let M be the manifold obtained by splitting $N(k)$ at $S(k)$. Then ∂M contains two copies of $S(k)$ and an annulus A coming from part of $\partial(N(k))$. Let J be the s.c.c. in ∂M that is the core of the annulus A. Since $S(k)$ is incompressible, $\partial M - J$ is incompressible. Since S is partially unknotted, ∂M is compressible.

Now, the closed surface $\hat{S}$ is incompressible in $\hat{M}$ if and only if the manifold obtained from M by adding a 2-handle along J has incompressible boundary. And this latter is precisely the conclusion of Theorem 2. $\square$

3. Remarks. I do not know how general it is for a knot in S^3 to have a partially unknotted, orientable, incompressible spanning surface. It is easy to construct knots with orientable, incompressible spanning surfaces that are *not* partially unknotted; however, I do not know of any knots that do not have the required type of spanning surface.

Probably the most interesting possibility is to obtain an appropriate generalization of Lemma 1. It is interesting to note that Lemma 1 and Theorem 2 are false in general, if M has incompressible boundary [**P**]. Furthermore, it seems to be difficult to find the appropriate conditions for the conclusions of Lemma 1 and Theorem 2 to hold if more than one 2-handle is added to M.

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