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# A CHARACTERIZATION OF ABSOLUTELY $C^*$-SMOOTH CONTINUA[1]

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ABSTRACT. A continuum $X$ is proven to be absolutely $C^*$-smooth if and only if each compactification $Y$ of the half line $[0, \infty)$ with remainder $X$ has the property that the space of all subcontinua of $Y$ is a compactification of the space of all subcontinua of $[0, \infty)$.

**1. Introduction.** After A. Lelek [**8**] introduced the notion of Class $W$ continua in 1972, several people have tried to determine which continua are contained in Class $W$ and have looked for different characterizations of Class $W$ continua. H. Cook [**1**] proved that all hereditarily indecomposable continua are in Class $W$. D. R. Read [**10**] showed that each chainable continuum is in Class $W$. G. A. Feuerbacher [**2**] proved that each nonplanar circle-like continuum is in Class $W$. B. Hughes [**3**] proved that Class $W$ contains all continua which have the covering property. J. Grispolakis and E. D. Tymchatyn proved that Class $W$ contains atriodic tree-like continua [**6**], compactifications of the half line $[0, \infty)$ with remainder a continuum in Class $W$ [**3**], and irreducible continua of type $\lambda$ that have each tranche as a tranche of cohesion and each nondegenerate tranche in Class $W$ [**3**]. They [**5**] have also proven that continua in Class $W$ are precisely those continua which are absolutely $C^*$-smooth or, equivalently, precisely those continua which have the covering property. This paper gives another characterization of Class $W$ continua by characterizing absolutely $C^*$-smooth continua.

**2. Definitions.** All *continua* are taken to be compact, connected, metric spaces. A mapping (continuous function) $f$ from a topological space $X$ onto a topological space $Y$ is *weakly confluent* if and only if each subcontinuum $K$ of $Y$ is the image of some component of $f^{-1}(K)$. A continuum $M$ is in *Class* $W$ if and only if all mappings from continua onto $M$ are weakly confluent. A continuum $X$ is *absolutely* $C^*$*-smooth* if and only if for each continuum $Y$ in which $X$ is embedded and for each sequence $\{X_i\}_{i=1}^{\infty}$ of subcontinua of $Y$ converging to $X$ it is true that $C(X) = \lim_{i\to\infty} C(X_i)$. If $Y$ is a compactification of a topological space $X$, then $Y - X$ is said to be the *remainder*.

**3. A characterization of absolutely $C^*$-smooth.** In [**3**] Grispolakis and Tymchatyn made good use of compactifications of the half line $[0, \infty)$ in obtaining continua in Class $W$. It is only natural to wonder whether it is possible to characterize Class $W$ (or, equivalently, absolutely $C^*$-smooth) continua in terms of remainders of the compactifications of the half line $[0, \infty)$.

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THEOREM. *A continuum $X$ is absolutely $C^*$-smooth if and only if each compactification $Y$ of the half line $[0, \infty)$ with remainder $X$ has the property that $C(Y)$ is a compactification of $C([0, \infty))$.*

PROOF. Suppose $X$ is a continuum which is absolutely $C^*$-smooth. Using that which is contained in a proof by Grispolakis and Tymchatyn [**3**, Theorem 3.5], there is a compactification $Y$ of $[0, \infty)$ which has $X$ as the remainder. For each natural number $n$, let $G_n$ be a finite open cover of $X$ with mesh less than $1/n$. For each natural number $n$, there is a natural number $m_n$ such that $[n, m_n]$ intersects each element in the cover $G_n$ since $[n, \infty) \cup X$ is a compactification of $[n, \infty)$. Now since $X$ is absolutely $C^*$-smooth and $X = \lim_{n \to \infty}[n, m_n]$, it must follow that $C(X) = \lim_{n \to \infty} C([n, m_n])$. This shows that $C(X)$ is a subset of the compactification of $C([0, \infty))$. Suppose $K$ is a subcontinuum of $Y$ which is not a subcontinuum of $X$. Notice that $K$ must contain at least one point of $[0, \infty)$. If $K$ contains no point of $X$, then $K$ is a subcontinuum of $[0, \infty)$ which places $K$ in $C([0, \infty))$. If $K$ contains a point of $X$, then $K$ must be $X \cup [p, \infty)$ for some point $p$ of $[0, \infty)$, which implies that $K = \lim_{n \to \infty}[p, n]$, and thus $K$ must be in the compactification of $C([0, \infty))$. This proves that $C(Y)$ is a subset of the compactification of $C([0, \infty))$. Since $C([0, \infty))$ is a subset of the compact set $C(Y)$, it is obvious that the compactification of $C([0, \infty))$ is a subset of $C(Y)$, which then establishes that $C(Y)$ is a compactification of $C([0, \infty))$.

Now suppose $X$ is a continuum such that each compactification $Y$ of $[0, \infty)$ which has $X$ as the remainder also has the property that $C(Y)$ is a compactification of $C([0, \infty))$. Further assume there is an embedding of $X$ in a continuum $Z$ such that there is a sequence $\{X_i\}_{i=1}^{\infty}$ of subcontinua of $Z$ and a subcontinuum $K$ of $X$ such that $X = \lim_{i \to \infty} X_i$ and $K$ is not in $\lim_{i \to \infty} C(X_i)$. It may be assumed at this point without any loss of generality that $Z$ is embedded in the Hilbert cube such that the first coordinate of each point in $Z$ is zero. There are finite open covers $G$ and $H$ of $K$ such that each element of $G \cup H$ contains a point of $K$, the closures of the elements of $H$ are subsets of elements of $G$, and there is a natural number $N$ having the property that $X_i$ does not have a subcontinuum of $G^*$ (the union of the elements of $G$) intersecting the closure of each element of $H$ for each $i > N$. Let $D$ denote an open subset of the Hilbert cube such that $\overline{H}^* \subset D$ and $\overline{D} \subseteq G^*$. Choose some element of $H$ and denote its closure by $A_1$. There is a maximum natural number $m$ such that infinitely many of the continua $\{X_i\}_{i>N}$ have subcontinua of $\overline{D}$ intersecting $A_1$ and the closures of $m - 1$ other sets in $H$. Of all the subcollections of $H$ containing $m$ sets, with one of its sets having closure equal to $A_1$, one of these subcollections has infinitely many of the continua $\{X_i\}_{i>N}$ possessing subcontinua of $\overline{D}$ intersecting the closures of the sets within this subcollection. The closures of the sets in this subcollection will be denoted by $\{A_i\}_{i=1}^{m}$. It may be assumed without loss of generality that each of the continua $\{X_i\}_{i>N}$ has at least one subcontinuum which is contained in $\overline{D}$ and intersects each of the sets $\{A_i\}_{i=1}^{m}$. Let $\{C_i\}_{i>0}$ denote subcontinua of $\{X_i\}_{i>N}$, respectively, such that each of $\{C_i\}_{i>0}$ is a subset of $\overline{D}$ and intersects each of $\{A_i\}_{i=1}^{m}$. It can be assumed that $\{C_i\}_{i>0}$ converges to a subcontinuum of $X$. Notice that no subcontinuum of the continua $\{X_i\}_{i>N}$ can intersect each of the sets $\{A_i\}_{i=1}^{m}$ and the closure of another different subset in $H$. This implies that there must be a positive number $\varepsilon_i$ for each $i > 0$ such that no continuum within

$\varepsilon_i$ of $X_{N+i}$ has a subcontinuum in $\overline{D}$ with points within $\varepsilon_i$ of each of $\{A_i\}_{i=1}^m$ and which intersects the closure of a set in $H$ different from each set in $\{A_i\}_{i=1}^m$. In order to see that this is correct, assume it is not true, in which case there exists a natural number $I$ such that for each natural number $n$ there is a continuum $E_n$ within $1/n$ of $X_{N+I}$ which has a subcontinuum $F_n$ in $\overline{D}$ with points within $1/n$ of the closed sets $\{A_i\}_{i=1}^m$ and which intersects the closure of a set in $H$ that is different from each of the sets $\{A_i\}_{i=1}^m$. Since $H$ is a finite set, one of the sets in $H$ has its closure $B$ intersected by infinitely many of the continua $\{F_i\}_{i>0}$; thus, the continua $\{E_i\}_{i>0}$ converge to $X_{N+I}$ with $\{F_i\}_{i>0}$ converging to a subcontinuum of both $X_{N+I}$ and $\overline{D}$ which intersects each of $\{A_i\}_{i=1}^m$ together with intersecting $B$, which is a contradiction of the agreement that no subcontinuum of both $X_{N+I}$ and $\overline{D}$ can intersect the closure of $m+1$ sets in $H$. This proves that positive numbers $\{\varepsilon_i\}_{i>0}$ as described above do indeed exist. Now a point $x_0$ is selected from $X$ which belongs to the subcontinuum of $X$ to which $\{C_i\}_{i>0}$ converges. Open balls $\{R_i\}_{i=1}^\infty$ of the Hilbert cube are selected such that each has $x_0$ as its center, and such that $R_n$ has radius less than $1/n$. Again without loss of generality it may be assumed that $R_i$ contains a point $p_i$ of $C_i$; otherwise, a subsequence of $\{C_i\}_{i>0}$ is selected and relabeled so that this is correct and the corresponding subsequence of $\{X_i\}_{i>N}$ which contains $\{C_i\}_{i>0}$ as subcontinua, respectively, is also selected and labeled as $\{X_i\}_{i>N}$. For each natural number $i$, let $S_i$ be a minimal finite collection of open balls covering $X_{N+i}$ with mesh less than the minimum of $\varepsilon_i$, $1/i$, and the distance from $C_i$ to the boundary of $D$. Let $S_i'$ denote the minimal subcollection of $S_i$ covering $C_i$. Define $q_1$ to be a point with first coordinate different from zero so that $q_1$ is a point of $R_1$ and of some open set in $S_1'$ that contains $p_1$. There is a piecewise-linear arc $\alpha_1$ which is a subset of $(S_1')^*$ such that $\alpha_1$ has as endpoints $q_1$ and some other point $r_1$, no point of $\alpha_1$ has its first coordinate equal to zero, and $\alpha_1$ passes through each open ball in $S_1'$. From any point of $C_1$ to any other point of $C_1$ there is a chain of open balls in $S_1'$. In particular, there is a point $z_1$ of $C_1$ in an open set $V$ in $S_1'$ that contains both $p_1$ and $q_1$. There is a chain of open balls in $S_1'$ from $z_1$ to some other point of $C_1$. This chain must contain $V$ as the first link; thus, a piecewise-linear arc can be constructed that begins at $q_1$ and runs to the last link of the chain. Another chain can now be considered whose first link is the same as the last link of the first chain, and the last link of the second is different from each link of the first chain. The arc $\alpha_1$ can continue to be constructed by passing it through this second chain. This process can be continued until $\alpha_1$ passes through each element of $S_1'$ in such a way that it has the above-mentioned properties. There is another piecewise-linear arc $\beta_1$ which is a subset of $S_1^*$ passing through each open ball in $S_1$ such that $\beta_1$ has $r_1$ and another point $s_1$ as endpoints, the point $r_1$ is the only point that $\alpha_1$ and $\beta_1$ have in common, and all points of $\beta_1$ have their first coordinates different from zero. Another piecewise-linear arc $\gamma_1$ runs from $s_1$ back through each open ball in $S_1$ in the reverse manner in which $\beta_1$ passed through the open balls in $S_1$ such that $\gamma_1$ is a subset of $S_1^*$, $\gamma_1$ has as endpoints $s_1$ and some other point $t_1$ which, together with $r_1$, belongs to a common element of $S_1$, the only point $\gamma_1$ has in common with $\alpha_1 \cup \beta_1$ is $s_1$, and all first coordinates of the points of $\gamma_1$ are different from zero. Now a piecewise-linear arc $\delta_1$ is selected within $(S_1')^*$ that runs from $t_1$ to another point $u_1$ which belongs to an open set in $S_1'$ that also contains $q_1$ such that $\delta_1$ passes through all the open balls in $S_1'$ in the reverse manner in which $\alpha_1$ passed through the elements $S_1'$, $\delta_1$ only has $t_1$ in common with $\alpha_1 \cup \beta_1 \cup \gamma_1$, and all points in $\delta_1$ have their first coordinates different from zero. An arc $\sigma_1$ within $R_1$ is run from $u_1$ to a point $q_2$ of $R_2 - (\alpha_1 \cup \beta_1 \cup \gamma_1 \cup \delta_1)$ such that $q_2$ is a point of some open set in $S_2'$ that contains $p_2$ and $u_1$ is the only point $\sigma_1$ has in common with $\alpha_1 \cup \beta_1 \cup \gamma_1 \cup \delta_1$. In general for $i > 1$, piecewise-linear arcs $\alpha_i, \beta_i, \gamma_i, \delta_i, \sigma_i$ and points $r_i, s_i, t_i, u_i, q_{i+1}$ are defined with respect to $S_i', S_i, R_{i+1}$, and $p_{i+1}$ in a way similar to the way $r_1, s_1, t_1, u_1, q_2$ were defined with respect to $S_1', S_1, R_2$, and $p_2$ above with the additional requirement that $q_i$ is the only point that $\alpha_i \cup \beta_i \cup \gamma_i \cup \delta_i \cup \sigma_i$ has in common with $\bigcup_{j=1}^{i-1}(\alpha_j \cup \beta_j \cup \gamma_j \cup \delta_j \cup \sigma_j)$. Notice that $\bigcup_{i=1}^{\infty}(\alpha_i \cup \beta_i \cup \gamma_i \cup \delta_i \cup \sigma_i)$ is a ray which has a compactification $Y$ with $X$ as the remainder. Since $C(Y)$ is a compactification of $C(\bigcup_{i=1}^{\infty}(\alpha_i \cup \beta_i \cup \gamma_i \cup \delta_i \cup \sigma_i))$, there is a sequence of intervals $\{I_i\}_{i>0}$ of the ray converging to $K$. For some natural number $n'$, $I_{n'}$ must be a subset of $D$ that intersects the closure of each element of $H$. For each $j$, there are points $l_j$ and $l_j'$ which have the property that $l_j$ is the last point on the arc $\alpha_j \cup \beta_j$ in the order $q_j$ to $s_j$ such that the interval in $\alpha_j \cup \beta_j$ from $q_j$ to $l_j$ is a subset of $\overline{D}$ and, similarly, $l_j'$ is the last point on the arc $\gamma_j \cup \delta_j$ in the order from $u_j$ to $s_j$ such that the interval in $\gamma_j \cup \delta_j$ from $u_j$ to $l_j'$ is a subset of $\overline{D}$. Notice that the interval from $l_j'$ to $l_{j+1}$ in the arc $\gamma_j \cup \delta_j \cup \sigma_j \cup \alpha_{j+1} \cup \beta_{j+1}$ cannot have $I_{n'}$ as a subset since the interval from $l_j'$ to $l_{j+1}$ intersects only $\{A_i\}_{i=1}^m$ of the closures of the sets in $H$. There must then exist a natural number $J$ such that $I_{n'}$ is a subset of $\beta_J \cup \gamma_J$ which now says that $\beta_J \cup \gamma_J$ is a continuum within $\varepsilon_J$ of $X_{N+J}$ with subcontinuum $I_{n'}$ in $\overline{D}$ with points within $\varepsilon_J$ of $\{A_i\}_{i=1}^m$ and intersecting the closure of another set (actually all of the other sets) in $H$ different from $\{A_i\}_{i=1}^m$. This contradiction proves that $X$ is absolutely $C^*$-smooth.

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