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COMPACT LIE GROUP ACTION AND EQUIVARIANT BORDISM

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ABSTRACT. Let G be a compact Lie group and H a compact Lie subgroup of G contained in the center of G with H^m the maximal subgroup in the center, H being H-boundary. Let $p_r\colon H^m \to H$ be the projection onto the rth factor and H_r be the rth factor of H^m. Let $\{L_r\}$ be a family of subgroups of G such that $L_r \cap H_r$ is nontrivial. Consider a G-manifold M^n with $p_r(G_x \cap H^m)$ trivial or containing L_r, for every x in M^n. The main result of the paper is that if $\forall x \in M^n$, $p_r(G_x \cap H^m)$ is trivial at least for one r, then M^n is a G-boundary.

1. Introduction. This paper is a sequel to [**3**–**5**]. Conner and Floyd [**1**] proved that if $\mathbf{Z}_2^k$ acts on a closed manifold M differentiably and without any fixed point, then M is a boundary. Stong [**7**] showed that if (M, θ) is a closed $\mathbf{Z}_2^k$-differential manifold without any stationary point, then (M, θ) is a $\mathbf{Z}_2^k$-boundary. In [**3**], we extended Stong's result for any finite abelian group of even order by proving the following. Let G be a finite abelian group of even order, (M, θ) a closed G-differential manifold and the elementary 2-group G_2 in G acts on M under θ without any stationary point. Then (M, θ) is a G-boundary. In [**4**], we inititated this problem for nonabelian groups S_3 and dihedral groups. In [**5**], we have extended the result of [**3**] for an arbitrary finite group with center of even order. One needs the elementary 2-group $G_2(C)$ of the center of G instead of G_2.

In the present note, we consider the action of a compact Lie group and prove that the induced action of the central elementary H-subgroup of G determines G-bordism. This gives the results of [**3**–**5**] in particular cases.[1]

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2. Preliminaries. Let H be a compact Lie group. If there exists an H-differential closed manifold (N, θ) such that the boundary $\partial N = H$ and the restriction of the action θ to H coincides with the operation in H, then we say that the compact Lie group H is H-boundary. Let G be a compact Lie group. By the *central elementary H-group* in G, we will mean the maximal subgroup H^n $(= H \times \cdots \times H$, n times) contained in the center of G.

Consider a compact Lie group G with H^n the central elementary H-group in G, H being H-boundary. Let us fix a point h_0 of H. Let $p_r\colon H^n \to H$ denote the projection onto the rth factor, $1 \le r \le n$. Let H_r denote the subgroup of H^n

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with $p_i(H_r) = H$, if $i = r$ and $p_i(H_r) = h_0$, if $i \neq r$. Consider a family $\{L_r\}$ of subgroups L_r of G such that $L_r \cap H_r$ is a nontrivial subgroup of H_r, $1 \leq r \leq n$. By an *$\{L_r\}$-type action* of G, we mean a differential action of G on a differential manifold such that for every x in M, $p_r(G_x \cap H^n)$ is either trivial or contains L_r $\forall r$, G_x being the isotropy group of x. A point x in M is said to be a *pseudo stationary point* if $p_r(G_x \cap H^n)$ is nontrivial $\forall r$, $1 \leq r \leq n$.

A family $\mathcal{F}$ in G is a collection of subgroups of G such that if $K \in \mathcal{F}$, then all the subgroups of K and all conjugates of K are in $\mathcal{F}$. Let $\mathcal{F}' \subset \mathcal{F}$ be families in G such that there exists an H-boundary subgroup H of G satisfying the following conditions:

(a) no nontrivial subgroup of H is contained in K $\forall K \in \mathcal{F} - \mathcal{F}'$,

(b) the intersection I of all the members of $\mathcal{F} - \mathcal{F}'$ is in $\mathcal{F} - \mathcal{F}'$,

(c) H is contained in the center.

We call such a pair $(\mathcal{F}, \mathcal{F}')$ of families an *admissible pair* in G with respect to $H \subset G$.

EXAMPLE 2.1. Consider a family $\{L_r\}$ of subgroups of G such that $L_r \cap H_r$ is a nontrivial subgroup of H_r. Let $\mathcal{F}_r^L$ denote the family of all subgroups K of G for which $p_j(K \cap H^n)$ is trivial at least for one $j = 1, \ldots, r$ and the nontrivial subgroups $p_j(K \cap H^n)$ of H_j contain L_j. Then $(\mathcal{F}_{r+1}^L, \mathcal{F}_r^L)$ is an admissible pair of families in G with respect to H_{r+1}, $0 \leq r \leq n$, $\mathcal{F}_0^L$ being the empty family.

3. $\mathcal{F}_n^L$-free action and G-bordism. The object of this section is to show that if (M, θ) is an $\mathcal{F}_n^L$-free closed G-manifold, then (M, θ) is G-boundary. Let $\mathfrak{N}_*(G; \mathcal{F}, \mathcal{F}')$ denote the $(\mathcal{F}, \mathcal{F}')$-free G-bordism group for a pair $(\mathcal{F}, \mathcal{F}')$ of families in G. For a given family $\mathcal{F}$ in G and a subgroup K of G, let $\mathcal{F}_K$ denote the smallest family in G containing all the subgroups $[S \cup P]$, $S \in \mathcal{F}$ and P a subgroup of K.

THEOREM 3.1. *If $(\mathcal{F}, \mathcal{F}')$ is an admissible pair of families in G with respect to a subgroup H, which is H-boundary, then the homomorphism*

$$\mathfrak{N}_*(G; \mathcal{F}, \mathcal{F}') \to \mathfrak{N}_*(G; \mathcal{F}_H, \mathcal{F}'_H)$$

induced by the inclusion map $(\mathcal{F}, \mathcal{F}') \to (\mathcal{F}_H, \mathcal{F}'_H)$ is the zero homomorphism.

PROOF. Let $[M, \theta]$ be in $\mathfrak{N}_n(G; \mathcal{F}, \mathcal{F}')$. Let F denote the fixed points set of I in M, I being the intersection of all the members of $\mathcal{F} - \mathcal{F}'$. Since $\mathcal{F} - \mathcal{F}'$ is invariant under conjugation, I is normal in G so that the action θ induces an action on F, which we once again denote by θ. Let ν be the normal bundle of the imbedding of F in the interior of M and $D(\nu)$ be the disc bundle with the action θ^* of G on $D(\nu)$ induced by the real vector bundle maps covering the action θ on F. Since F is the fixed points set of I, no nontrivial subgroup of H is contained in K $\forall K \in \mathcal{F} - \mathcal{F}'$, no point of F will be fixed by the subgroup $[I \cup P]$, P being a nontrivial subgroup of H, so that H will act freely on F and hence on $D(\nu)$. Let $F' = F/H$ and $D'(\nu) = D(\nu)/H$. The actions θ and θ^* on F and on $D(\nu)$ induce actions θ' and $\theta^{*\prime}$ on F' and $D'(\nu)$ respectively, because H is contained in the center. Since H acts freely on F and $D(\nu)$, the quotient maps $\xi_1\colon F \to F'$ and $\xi_2\colon D(\nu) \to D'(\nu)$ are principal H-bundles. Since H is H-boundary, there exists an H-differential closed manifold $(N, \emptyset)$ such that the boundary $\partial N = H$ and the restriction of the action $\emptyset$ to H coincides with the operation in H. Consider the fibre bundles $\tilde{\xi}_1 = \xi_1[N]$ and $\tilde{\xi}_2 = \xi_2[N]$ associated to the principal H-bundles

ξ_1 and ξ_2 respectively. The total space E_1 of $\tilde{\xi}_1$ is given by $E_1 = (F \times N)/H$, where the action of H on $F \times N$ is given by $h(m,t) = (mh, h^{-1}t)$, $h \in H$ and $(m,t) \in F \times N$. Also the boundary ∂E_1 is diffeomorphic to $(F \times H)/H$. Let us take a fixed point $\tilde{h}$ of H. Define a map η: $(F \times H)/H \to F$ as $\eta([m,h]) = m\overline{h}$, where $h = \overline{h}\tilde{h}$. Clearly η is a diffeomorphism. Let us define an action ψ_1 of G on E_1 as $g[m,t] = [mg,t]$. Then the diffeomorphism η preserves the H-action. Thus E_1 is a G-manifold with ∂E_1 being equivariantly diffeomorphic to F. Similarly the total space E_2 of $\tilde{\xi}_2$ is $(D(\nu) \times N)/H$, where the action of H on $D(\nu) \times N$ is given by $h(m,t) = (mh, h^{-1}t)$. Consider the action ψ_2 of G on E_2 as $g[m,t] = [mg,t]$. Let α: $E_2 \to E_1$ be the map induced from ν': $D'(\nu) \to F'$ by going to the fibre bundles; one has the commutative diagram:

$$\begin{array}{lccc} \xi_2(N): & E_2 & \to & D'(\nu) \\ & \downarrow\alpha & & \downarrow\nu' \\ \xi_1(N): & E_1 & \to & F' \end{array}$$

Also $\alpha^{-1}(\partial E_1)$ is diffeomorphic to $D(\nu)$ and the action ψ_2 on $\alpha^{-1}(\partial E_1)$ is isomorphic to the action θ^* on $D(\nu)$. Consider

$$W = (M \times [0,1]) \cup E_2/\sim$$

where $\sim$ is the equivalence relation in W obtained by identifying $D(\nu) \times \{1\}$ with $\alpha^{-1}(\partial E_1)$. Let the action Φ of G on W be defined by $\Phi|M \times [0,1] = \theta \times 1$ and $\Phi|E_2 = \psi_2$. Take V to be

$$(\partial M \times [0,1]) \cup (M \times \{1\} - (D(\nu) \times \{1\})^\circ) \cup (\partial E_2 - (\alpha^{-1}(\partial E_1))^\circ),$$

where $^\circ$ denotes the interior operator. Since I is the intersection of all members of $\mathcal{F} - \mathcal{F}'$, V will be $(\mathcal{F}'_H, \mathcal{F}'_H)$-free. Also W is $(\mathcal{F}_H, \mathcal{F}'_H)$-free and ∂W is diffeomorphic to $M \cup V$ identifying ∂V with ∂M. This shows that $[M, \theta]$ is zero in $\mathfrak{N}_*(G; \mathcal{F}_H, \mathcal{F}'_H)$. □

Let $\mathfrak{A}$ denote the family of all subgroups of G. Following the notations of Example 2.1 and using Theorem 3.1 we get the following

COROLLARY 3.2. *For every r, $0 \le r \le n$, the homomorphism $\mathfrak{N}_*(G; \mathcal{F}^L_{r+1}, \mathcal{F}^L_r) \to \mathfrak{N}_*(G; \mathfrak{A}, \mathcal{F}^L_r)$ induced from the inclusion map $(\mathcal{F}^L_{r+1}, \mathcal{F}^L_r) \to (\mathfrak{A}, \mathcal{F}^L_r)$ is the zero one.*

PROOF. Since $(\mathcal{F}^L_{r+1}, \mathcal{F}^L_r)$ is an admissible pair of families with respect to the subgroup H_{r+1}, $0 \le r < n$, and $(\mathcal{F}^L_r)_{H_{r+1}} = \mathcal{F}^L_r$, Theorem 3.1 gives the corollary. □

COROLLARY 3.3. *Let M be a closed G-manifold with $\{L_r\}$-type of action for some family $\{L_r\}$ of subgroups of G such that $L_r \cap H_r$ is nontrivial. If M does not have any pseudo stationary point, then M is a G-boundary.*

PROOF. It is enough to show that the homomorphism $\mathfrak{N}_*(G; \mathcal{F}^L_n) \to \mathfrak{N}_*(G; \mathfrak{A})$ induced from the inclusion map $\mathcal{F}^L_n \to \mathfrak{A}$ is the zero one. By Corollary 3.2 and the exact bordism sequence for the triple $(\mathfrak{A}, \mathcal{F}^L_{r+1}, \mathcal{F}^L_r)$, one gets that $j_*: \mathfrak{N}(G; \mathfrak{A}, \mathcal{F}^L_r) - \mathfrak{N}_*(G; \mathfrak{A}, \mathcal{F}^L_{r+1})$ is a monomorphism, j: $(\mathfrak{A}, \mathcal{F}^L_r) \to (\mathfrak{A}, \mathcal{F}^L_{r+1})$ is the inclusion map. Therefore the composite

$$\mathfrak{N}_*(G; \mathfrak{A}, \mathcal{F}^L_0) \to \mathfrak{N}_*(G; \mathfrak{A}, \mathcal{F}^L_1) \to \cdots \to \mathfrak{N}_*(G; \mathfrak{A}, \mathcal{F}^L_n)$$

is a monomorphism and hence by the bordism exact sequence of the triple $(\mathfrak{A}, \mathcal{F}_n^L, \mathcal{F}_0^L)$, one gets that $\mathfrak{N}_*(G; \mathcal{F}_n^L, \mathcal{F}_0^L) \to \mathfrak{N}_*(G; \mathfrak{A}, \mathcal{F}_0^L)$ is the zero homomorphism. This completes the proof, since $\mathcal{F}_0^L$ is empty. □

REMARK 3.4. Taking G to be a finite abelian group of even order and H to be $\mathbf{Z}_2$, one gets Corollary 3.7 of [**3**]. Considering G to be a finite group with center of even order and H to be $\mathbf{Z}_2$, one gets Corollary 3.3 of [**5**].

REMARK 3.5. The case $H = \mathbf{Z}_2$ can also be obtained in a simpler way using the technique used in the Appendix of [**2**]. The case $G =$ finite group and $H = \mathbf{Z}_2$, has been obtained by Kosniowski [**6**] using the concept of slices.

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