COMPACT LIE GROUP ACTION
AND EQUIVARIANT BORDISM

SHABD SHARAN KHARE

ABSTRACT. Let $G$ be a compact Lie group and $H$ a compact Lie subgroup of $G$ contained in the center of $G$ with $H^m$ the maximal subgroup in the center, $H$ being $H$-boundary. Let $p_r: H^m \to H$ be the projection onto the $r$th factor and $H_r$ be the $r$th factor of $H^m$. Let $\{L_r\}$ be a family of subgroups of $G$ such that $L_r \cap H_r$ is nontrivial. Consider a $G$-manifold $M^n$ with $p_r(G_x \cap H^m)$ trivial or containing $L_r$, for every $x$ in $M^n$. The main result of the paper is that if $\forall x \in M^n$, $p_r(G_x \cap H^m)$ is trivial at least for one $r$, then $M^n$ is a $G$-boundary.

1. Introduction. This paper is a sequel to [3–5]. Conner and Floyd [1] proved that if $\mathbb{Z}_2$ acts on a closed manifold $M$ differentiably and without any fixed point, then $M$ is a boundary. Stong [7] showed that if $(M, \theta)$ is a closed $\mathbb{Z}_2^k$-differential manifold without any stationary point, then $(M, \theta)$ is a $\mathbb{Z}_2^k$-boundary. In [3], we extended Stong's result for any finite abelian group of even order by proving the following. Let $G$ be a finite abelian group of even order, $(M, \theta)$ a closed $G$-differential manifold and the elementary 2-group $G_2$ in $G$ acts on $M$ under $\theta$ without any stationary point. Then $(M, \theta)$ is a $G$-boundary. In [4], we initiated this problem for nonabelian groups $S_3$ and dihedral groups. In [5], we have extended the result of [3] for an arbitrary finite group with center of even order. One needs the elementary 2-group $G_2(C)$ of the center of $G$ instead of $G_2$.

In the present note, we consider the action of a compact Lie group and prove that the induced action of the central elementary $H$-subgroup of $G$ determines $G$-bordism. This gives the results of [3–5] in particular cases.\footnote{The author was partially supported by DAE Grant.}

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2. Preliminaries. Let $H$ be a compact Lie group. If there exists an $H$-differential closed manifold $(N, \theta)$ such that the boundary $\partial N = H$ and the restriction of the action $\theta$ to $H$ coincides with the operation in $H$, then we say that the compact Lie group $H$ is $H$-boundary. Let $G$ be a compact Lie group. By the central elementary $H$-group in $G$, we will mean the maximal subgroup $H^n (= H \times \cdots \times H$, $n$ times) contained in the center of $G$.

Consider a compact Lie group $G$ with $H^n$ the central elementary $H$-group in $G$, $H$ being $H$-boundary. Let us fix a point $h_0$ of $H$. Let $p_r: H^n \to H$ denote the projection onto the $r$th factor, $1 \leq r \leq n$. Let $H_r$ denote the subgroup of $H^n$...
with \( p_i(H_r) = H \), if \( i = r \) and \( p_i(H_r) = h_0 \), if \( i \neq r \). Consider a family \( \{L_r\} \) of subgroups \( L_r \) of \( G \) such that \( L_r \cap H_r \) is a nontrivial subgroup of \( H_r \), \( 1 \leq r \leq n \). By an \( \{L_r\}\)-type action of \( G \), we mean a differential action of \( G \) on a differential manifold such that for every \( x \in M \), \( p_i(G_x \cap H^n) \) is either trivial or contains \( L_r \) \( \forall r \), \( G_x \) being the isotropy group of \( x \). A point \( x \) in \( M \) is said to be a pseudo stationary point if \( p_i(G_x \cap H^n) \) is nontrivial \( \forall r, 1 \leq r \leq n \).

A family \( \mathcal{F} \) in \( G \) is a collection of subgroups of \( G \) such that if \( K \in \mathcal{F} \), then all the subgroups of \( K \) and all conjugates of \( K \) are in \( \mathcal{F} \). Let \( \mathcal{F}' \subset \mathcal{F} \) be families in \( G \) such that there exists an \( H \)-boundary subgroup \( H \) of \( G \) satisfying the following conditions:

(a) no nontrivial subgroup of \( H \) is contained in \( K \), \( \forall K \in \mathcal{F} - \mathcal{F}' \),

(b) the intersection \( I \) of all the members of \( \mathcal{F} - \mathcal{F}' \) is in \( \mathcal{F} - \mathcal{F}' \),

(c) \( H \) is contained in the center.

We call such a pair \( (\mathcal{F}, \mathcal{F}') \) of families an admissible pair in \( G \) with respect to \( H \subset G \).

**EXAMPLE 2.1.** Consider a family \( \{L_r\} \) of subgroups of \( G \) such that \( L_r \cap H_r \) is a nontrivial subgroup of \( H_r \). Let \( \mathcal{K} \) denote the family of all subgroups \( K \) of \( G \) for which \( p_j(K \cap H^n) \) is trivial at least for one \( j = 1, \ldots, r \) and the nontrivial subgroups \( p_j(K \cap H^n) \) of \( H_j \) contain \( L_j \). Then \( (\mathcal{F}, \mathcal{F}') \) is an admissible pair of families in \( G \) with respect to \( H_{r+1}, 0 \leq r \leq n, \mathcal{F}' \) being the empty family.

**3. \( \mathcal{F}_n \)-free action and \( G \)-bordism.** The object of this section is to show that if \( (M, \theta) \) is an \( \mathcal{F}_n \)-free closed \( G \)-manifold, then \( (M, \theta) \) is \( G \)-boundary. Let \( \mathfrak{N}_\ast(G; \mathcal{F}, \mathcal{F}') \) denote the \( (\mathcal{F}, \mathcal{F}') \)-free \( G \)-bordism group for a pair \( (\mathcal{F}, \mathcal{F}') \) of families in \( G \). For a given family \( \mathcal{F} \) in \( G \) and a subgroup \( K \) of \( G \), let \( \mathcal{F}_K \) denote the smallest family in \( G \) containing all the subgroups \( [S \cup P], S \in \mathcal{F} \) and \( P \) a subgroup of \( K \).

**THEOREM 3.1.** If \( (\mathcal{F}, \mathcal{F}') \) is an admissible pair of families in \( G \) with respect to a subgroup \( H \), which is \( H \)-boundary, then the homomorphism

\[
\mathfrak{N}_\ast(G; \mathcal{F}, \mathcal{F}') \to \mathfrak{N}_\ast(G; \mathcal{F}_H, \mathcal{F}'_H)
\]

induced by the inclusion map \( (\mathcal{F}, \mathcal{F}') \to (\mathcal{F}_H, \mathcal{F}'_H) \) is the zero homomorphism.

**PROOF.** Let \([M, \theta]\) be in \( \mathfrak{N}_\ast(G; \mathcal{F}, \mathcal{F}') \). Let \( F \) denote the fixed points set of \( I \) in \( M \), \( I \) being the intersection of all the members of \( \mathcal{F} - \mathcal{F}' \). Since \( \mathcal{F} - \mathcal{F}' \) is invariant under conjugation, \( I \) is normal in \( G \) so that the action \( \theta \) induces an action on \( F \), which we once again denote by \( \theta \). Let \( \nu \) be the normal bundle of the imbedding of \( F \) in the interior of \( M \) and \( D(\nu) \) be the disc bundle with the action \( \nu \ast \) on \( D(\nu) \) induced by the real vector bundle maps covering the action \( \theta \) on \( F \). Since \( F \) is the fixed points set of \( I \), no nontrivial subgroup of \( H \) is contained in \( K \), \( \forall K \in \mathcal{F} - \mathcal{F}' \), no point of \( F \) will be fixed by the subgroup \([I \cup P], P \) being a nontrivial subgroup of \( H \), so that \( H \) will act freely on \( F \) and hence on \( D(\nu) \). Let \( F' = F/H \) and \( D'(\nu) = D(\nu)/H \). The actions \( \theta \) and \( \theta \ast \) on \( F \) and on \( D(\nu) \) induce actions \( \theta' \) and \( \theta \ast' \) on \( F' \) and \( D'(\nu) \) respectively, because \( H \) is contained in the center. Since \( H \) acts freely on \( F \) and \( D(\nu) \), the quotient maps \( \xi_1: F \to F' \) and \( \xi_2: D(\nu) \to D'(\nu) \) are principal \( H \)-bundles. Since \( H \) is \( H \)-boundary, there exists an \( H \)-differential closed manifold \( (N, \theta) \) such that the boundary \( \partial N = H \) and the restriction of the action \( \theta \) to \( H \) coincides with the operation in \( H \). Consider the fibre bundles \( \tilde{\xi}_1 = \xi_1[N] \) and \( \tilde{\xi}_2 = \xi_2[N] \) associated to the principal \( H \)-bundles.
The total space $E_1$ of $\xi_1$ is given by $E_1 = (F \times N)/H$, where the action of $H$ on $F \times N$ is given by $h(m, t) = (mh, h^{-1}t)$, $h \in H$ and $(m, t) \in F \times N$. Also the boundary $\partial E_1$ is diffeomorphic to $(F \times H)/H$. Let us take a fixed point $\tilde{h}$ of $H$. Define a map $\eta: (F \times H)/H \to F$ as $\eta([m, h]) = mh$, where $h = \tilde{h}h$. Clearly $\eta$ is a diffeomorphism. Let us define an action $\psi_1$ of $G$ on $E_1$ as $g[m, t] = [mg, t]$. Then the diffeomorphism $\eta$ preserves the $H$-action. Thus $E_1$ is a $G$-manifold with $\partial E_1$ being equivariantly diffeomorphic to $F$. Similarly the total space $E_2$ of $\xi_2$ is $(D(v) \times N)/H$, where the action of $H$ on $D(v) \times N$ is given by $h(m, t) = (mh, h^{-1}t)$. Consider the action $\psi_2$ of $G$ on $E_2$ as $g[m, t] = [mg, t]$. Let $\alpha: E_2 \to E_1$ be the map induced from $\nu: D(v) \to F'$ by going to the fibre bundles; one has the commutative diagram:

$$
\begin{array}{ccc}
\xi_2(N): & E_2 & \to & D'(\nu) \\
\downarrow \alpha & \downarrow \nu \\
\xi_1(N): & E_1 & \to & F'
\end{array}
$$

Also $\alpha^{-1}(\partial E_1)$ is diffeomorphic to $D(\nu)$ and the action $\psi_2$ on $\alpha^{-1}(\partial E_1)$ is isomorphic to the action $\theta^\ast$ on $D(\nu)$. Consider

$$W = (M \times [0, 1]) \cup E_2 / \sim$$

where $\sim$ is the equivalence relation in $W$ obtained by identifying $D(\nu) \times \{1\}$ with $\alpha^{-1}(\partial E_1)$. Let the action $\Phi$ of $G$ on $W$ be defined by $\Phi|M \times [0, 1] = \theta \times 1$ and $\Phi|E_2 = \psi_2$. Take $V$ to be

$$(\partial M \times [0, 1]) \cup (M \times \{1\} - (D(\nu) \times \{1\})^\circ) \cup (\partial E_2 - (\alpha^{-1}(\partial E_1))^\circ),$$

where $^\circ$ denotes the interior operator. Since $I$ is the intersection of all members of $\mathcal{F} - \mathcal{F}_i$, $V$ will be $(\mathcal{F}_H, \mathcal{F}_H^\ast)$-free. Also $W$ is $(\mathcal{F}_H, \mathcal{F}_H^\ast)$-free and $\partial W$ is diffeomorphic to $M \cup V$ identifying $\partial V$ with $\partial M$. This shows that $[M, \theta]$ is zero in $\mathcal{W}_\ast(G; \mathcal{F}_H, \mathcal{F}_H^\ast)$. □

Let $\mathfrak{A}$ denote the family of all subgroups of $G$. Following the notations of Example 2.1 and using Theorem 3.1 we get the following

**Corollary 3.2.** For every $r, 0 \leq r \leq n$, the homomorphism $\mathcal{W}_\ast(G; \mathcal{F}_{r+1}, \mathcal{F}_r^\ast) \to \mathcal{W}_\ast(G; \mathfrak{A}, \mathcal{F}_r^\ast)$ induced from the inclusion map $(\mathcal{F}_{r+1}, \mathcal{F}_r^\ast) \to (\mathfrak{A}, \mathcal{F}_r^\ast)$ is the zero one.

**Proof.** Since $(\mathcal{F}_{r+1}, \mathcal{F}_r^\ast)$ is an admissible pair of families with respect to the subgroup $H_{r+1}$, $0 \leq r < n$, and $(\mathcal{F}_r^\ast)_{H_{r+1}} = \mathcal{F}_r^\ast$, Theorem 3.1 gives the corollary. □

**Corollary 3.3.** Let $M$ be a closed $G$-manifold with $\{L_r\}$-type of action for some family $\{L_r\}$ of subgroups of $G$ such that $L_r \cap H_r$ is nontrivial. If $M$ does not have any pseudo stationary point, then $M$ is a $G$-boundary.

**Proof.** It is enough to show that the homomorphism $\mathcal{W}_\ast(G; \mathcal{F}_n^\ast) \to \mathcal{W}_\ast(G; \mathfrak{A})$ induced from the inclusion map $\mathcal{F}_n^\ast \to \mathfrak{A}$ is the zero one. By Corollary 3.2 and the exact bordism sequence for the triple $(\mathfrak{A}, \mathcal{F}_n^\ast, \mathcal{F}_r^\ast)$, one gets that $j_*: \mathcal{W}_\ast(G; \mathfrak{A}, \mathcal{F}_r^\ast) \to \mathcal{W}_\ast(G; \mathfrak{A}, \mathcal{F}_r^\ast)_{H_{r+1}}$ is a monomorphism, $j: (\mathfrak{A}, \mathcal{F}_r^\ast) \to (\mathfrak{A}, \mathcal{F}_r^\ast)_{H_{r+1}}$ is the inclusion map. Therefore the composite

$$\mathcal{W}_\ast(G; \mathfrak{A}, \mathcal{F}_0^\ast) \to \mathcal{W}_\ast(G; \mathfrak{A}, \mathcal{F}_1^\ast) \to \cdots \to \mathcal{W}_\ast(G; \mathfrak{A}, \mathcal{F}_n^\ast)$$
is a monomorphism and hence by the bordism exact sequence of the triple
\((\mathcal{A}, \mathcal{F}_n^L, \mathcal{F}_0^L)\), one gets that \(\Omega_\ast(G; \mathcal{F}_n^L, \mathcal{F}_0^L) \rightarrow \Omega_\ast(G; \mathcal{A}, \mathcal{F}_0^L)\) is the zero homomorphism. This completes the proof, since \(\mathcal{F}_0^L\) is empty. \(\square\)

**Remark 3.4.** Taking \(G\) to be a finite abelian group of even order and \(H\) to be \(\mathbb{Z}_2\), one gets Corollary 3.7 of [3]. Considering \(G\) to be a finite group with center of even order and \(H\) to be \(\mathbb{Z}_2\), one gets Corollary 3.3 of [5].

**Remark 3.5.** The case \(H = \mathbb{Z}_2\) can also be obtained in a simpler way using the technique used in the Appendix of [2]. The case \(G = \text{finite group and } H = \mathbb{Z}_2\), has been obtained by Kosniowski [6] using the concept of slices.

**References**


Department of Mathematics, North-Eastern Hill University, Bijni Campus, Bhagyakul Road, Laitumkhrah, Shillong 793 003, Meghalaya India