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ON 2-KNOT GROUPS WITH ABELIAN COMMUTATOR SUBGROUPS

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ABSTRACT. In this paper it is shown that if the commutator subgroup of a 2-knot group is abelian, then it is isomorphic to $Z \oplus Z \oplus Z$, Z_α, $Z[1/2]$ or $Z[1/2] \oplus Z_5$, where α is an odd integer and $Z[1/2]$ is the additive group of the dyadic rationals.

1. Introduction. An *n-knot* K is a smooth submanifold of a homotopy $(n+2)$-sphere Σ^{n+2} which is homeomorphic to the n-sphere S^n. The fundamental group of the complement $\Sigma^{n+2} - K$ is called the *group* of K.

In **[11]**, Levine completely determined 2-knot groups with finitely generated abelian commutator subgroups. He showed that a finitely generated abelian group is isomorphic to the commutator subgroup of a 2-knot group if and only if it is $Z \oplus Z \oplus Z$ or Z_α, where α is an odd integer. In this paper, we will consider 2-knot groups whose commutator subgroups are abelian and not finitely generated, and the following result will be obtained.

THEOREM. *Let G be a 2-knot group. If the commutator subgroup G' is abelian and not finitely generated, then it is isomorphic to $Z[1/2]$ or $Z[1/2] \oplus Z_5$, where $Z[1/2]$ is the additive group of the dyadic rationals.*

There exists a 2-knot group whose commutator subgroup is isomorphic to $Z[1/2]$ **[8]**. On the other hand, it still remains open whether $Z[1/2] \oplus Z_5$ can be realized as the commutator subgroup of a 2-knot group. If there exists such a 2-knot, its group would be presented by

$$(*) \qquad \langle a, b, t \colon t^{-1}at = a^2, t^{-1}bt = b^{-1}, b^5, [a, b] \rangle.$$

Question. Does there exist a 2-knot group whose commutator subgroup is isomorphic to $Z[1/2] \oplus Z_5$?

2. Preliminaries. Let X be an abelian group. We denote the torsion part of X by $T(X)$ and denote $X/T(X)$ by $F(X)$. By the *torsion-free rank* $r(X)$ of X, we mean the dimension of the vector space $X \otimes Q$, where Q is the field of rational numbers. Similarly, for a prime p, the *p-rank* $r_p(X)$ of X is defined by the dimension of the vector space $T(X) \otimes F_p$ over F_p, where F_p is the field of order p.

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The second exterior power $\wedge^2 X$ of X is defined by $X \otimes X/D$, where $D = \langle x \otimes x: x \in X \rangle$ [**1**]. We will denote an element $x_1 \otimes x_2 + D$ of $\wedge^2 X$ by $x_1 \wedge x_2$.

Let $1 \to R \to F \to X \to 1$ be a presentation of X. By [**7**], the second homology group $H_2(X)$ of X is isomorphic to $R \cap [F, F]/[R, F]$. Since X is abelian, we have $[F, F] \subset R$, and so $H_2(X) \cong [F, F]/[R, F]$. Therefore, $\wedge^2 X$ and $H_2(X)$ are isomorphic by the mapping $f(x_1 \wedge x_2) = [\tilde{x}_1, \tilde{x}_2][R, F]$, where $\tilde{x}_1, \tilde{x}_2 \in F$ represent $x_1, x_2 \in X$, respectively [**16**].

For an n-knot K, let G be the group of K and $\tilde{E}$ the infinite cyclic covering of the complement $\Sigma^{n+2} - K$. Then the homologies $H_q(\tilde{E})$ are left modules over the ring $\Lambda = Z[t, t^{-1}]$, where t is a generator of the group of covering transformations [**10**]. Since $\pi_1(\tilde{E}) \cong G'$, these Λ-module structures induce those on the integral homology groups $H_q(G')$, $q = 1, 2$. More directly, we may consider that $H_1(G)$ operates on $H_q(G')$ by conjugation [**11**]. (This is well defined since inner automorphisms of G' induce the identity on homology.) If G' is abelian, then the left Λ-module structure on $\wedge^2 G'$ is given by $t(g_1 \wedge g_2) = tg_1 \wedge tg_2$ for any elements g_1 and g_2 of G'.

In this paper, we will consider only modules over the commutative ring Λ. Thus we may not distinguish between left and right Λ-modules.

Let $H = \langle S : R \rangle$ be a group, and let A and B be subgroups of H with an isomorphism ϕ: $A \to B$. The *HNN extension of H relative to A, B, and ϕ* [**6**, **12**] is defined by the group

$$G = \langle S, t: R, t^{-1}at = \phi(a), a \in A \rangle.$$

The group H is called the *base* of G, and A and B are called the *associated subgroups*. We denote the group G by $\mathrm{HNN}(H, A, B, \phi)$. We can see that the obvious homomorphism of H to G is a monomorphism [**12**]. Hence, we can consider H as a subgroup of G.

From [**4**], we can easily see that any n-knot group ($n \geqslant 1$) has a finitely presented base and finitely generated associated subgroups.

3. Lemmas. Throughout this section, let G be a 2-knot group with abelian commutator subgroup G'. From §2, G has a finitely presented base H. Since H is contained in G', it is a finitely generated abelian group.

PROPOSITION 3.1. *Let $G = \mathrm{HNN}(H, A, B, \phi)$. Then $H = A$ or $H = B$.*

PROOF. Let M be the subgroup of G generated by elements $\{h, t^{-1}ht: h \in H\}$. Then M can be considered as a free product of $t^{-1}Ht$ and H with amalgamated subgroups $t^{-1}At$ and B. Since M is contained in G', M is abelian. Hence M is indecomposable with respect to amalgamated product [**13**, p. 227]. Thus, we have $t^{-1}Ht = t^{-1}At$ or $H = B$. The proof is completed.

It is easily seen that G' is finitely generated if and only if $H = A = B$. From now on, we assume that G' is not finitely generated. Furthermore, without loss of generalities, we may assume that $H = A$ and $H \neq B$. Then G' is a union of an ascending chain of subgroups

$$H < tHt^{-1} < t^2Ht^{-2} < \cdots,$$

and the torsion-free rank of G' is equal to the torsion-free rank of H. A Λ-module presentation of G' is given by

(1)
$$\left\langle a_1,\dots,a_n, b_1,\dots,b_m \colon ta_i = \sum_{j=1}^{n} \alpha_{ij}a_j + \sum_{j=1}^{m} \beta_{ij}b_j,\right.$$
$$\left. i = 1,\dots,n,\ tb_k = \sum_{j=1}^{m} \gamma_{kj}b_j,\ \lambda_k b_k = 0,\ k = 1,\dots,m \right\rangle,$$

where $n = r(G')$; $\alpha_{ij}, \beta_{ij}, \gamma_{kj}, \lambda_k \in Z$; and $\lambda_k \neq 0$.

Since $\phi(T(H)) \subset T(B) \subset T(H)$, and $T(H)$ is finite, we have $T(B) = T(H)$. Therefore we see that $F(H) \supsetneqq F(B)$. Thus we get $\det(\alpha_{ij}) \neq \pm 1, 0$. Furthermore, since $T(G') = T(H)$ is finite, it follows from [**3**, Proposition 100.2] that G' splits as an abelian group, i.e., $G' \cong F(G') \oplus T(G')$.

PROPOSITION 3.2. *$F(G')$ and $T(G')$ are isomorphic to commutator subgroups of certain n-knot groups* ($n \geqslant 3$).

PROOF. See [**5** or **11**].

For a Λ-module M, $\overline{M}$ denotes the conjugate module of M. That is, $\overline{M}$ is additively isomorphic to M, but t acts on $\overline{M}$ identically with t^{-1} on M. Levine proved the following

LEMMA 3.3 [**10**, p. 10; **11**, p. 251]. *The second homology group $H_2(G')$ is a Λ-homomorphic image of* $\operatorname{Ext}^1_\Lambda(\overline{G'}, \Lambda)$.

From definition of $F(G')$ and the presentation (1) of G', we have a Λ-module presentation of $F(G')$

$$\left\langle a_1,\dots,a_n \colon ta_i = \sum_{j=1}^{n} \alpha_{ij}a_j,\ i = 1,\dots,n \right\rangle. \tag{2}$$

Let $\tilde{F}(G')$ be the Λ-module presented by

$$\left\langle a_1,\dots,a_n \colon ta_i = \sum_{j=1}^{n} \alpha_{ji}a_j,\ i = 1,\dots,n \right\rangle. \tag{3}$$

Then we will show the following

LEMMA 3.4. $\operatorname{Ext}^1_\Lambda(\overline{G'}, \Lambda)$ *is Λ-isomorphic to* $\overline{\tilde{F}(G')}$.

PROOF. From (2), a free resolution of $F(G')$ as a Δ-module is given by

$$0 \to A_n \xrightarrow{t-\phi} A_n \to F(G') \to 0,$$

where A_n is the free Λ-module with a basis $\{a_1,\dots,a_n\}$, and ϕ is the Λ-endomorphism of A_n defined by

$$\phi(a_i) = \sum_{j=1}^{n} \alpha_{ij}a_j, \qquad i = 1,\dots,n.$$

Taking the dual $\operatorname{Hom}_\Lambda(\cdot, \Lambda)$ of this sequence, from [**14**, p. 60 and **10**, p. 3] we obtain the exact sequence of Λ-modules

$$0 \to \mathrm{Hom}_\Lambda(A_n, \Lambda) \overset{t-\phi^*}{\to} \mathrm{Hom}_\Lambda(A_n, \Lambda) \to \mathrm{Ext}^1_\Lambda(F(G'), \Lambda) \to 0,$$

where ϕ^* is the dual of ϕ.

By [**14**], $\mathrm{Hom}_\Lambda(A_n, \Lambda)$ is the free Λ-module with a basis $\{h_1,\ldots,h_n\}$, where $h_i(a_j) = \delta_{ij}$ (the Kronecker symbol), $i, j = 1,\ldots,n$. Therefore, a Λ-homomorphism f of A_n to $\mathrm{Hom}_\Lambda(A_n, \Lambda)$ given by $f(a_i) = h_i$ is a Λ-isomorphism. Under this isomorphism, $t - \phi^*$ induces the Λ-endomorphism $t - \tilde{\phi}$ of A_n, where $\tilde{\phi}(a_i) = \sum_{j=1}^n \alpha_{ji} a_j$, $i = 1,\ldots,n$ (cf. [**1**, p. 344]). Hence, we have the exact sequence of Λ-modules

$$0 \to A_n \overset{t-\tilde{\phi}}{\to} A_n \to \mathrm{Ext}^1_\Lambda(F(G'), \Lambda) \to 0.$$

Thus, $\mathrm{Ext}^1_\Lambda(F(G'), \Lambda)$ is Λ-isomorphic to $\tilde{F}(G')$. Since

$$\mathrm{Ext}^1_\Lambda(\overline{G'}, \Lambda) \underset{\Lambda}{\cong} \overline{\mathrm{Ext}^1_\Lambda(G', \Lambda)} \underset{\Lambda}{\cong} \overline{\mathrm{Ext}^1_\Lambda(F(G'), \Lambda)}$$

[**10**], this completes the proof.

REMARK. In a sense, Lemma 3.4 is a generalization of [**11**, p. 257].

LEMMA 3.5. *The torsion-free rank of $H_2(F(G'))$ is equal to $n(n-1)/2$, where $n = r(F(G'))$.*

PROOF. Let $k = n(n-1)/2$. If $n = 1$, then we have $\wedge^2 F(G') = 0$. For $n \geqslant 2$, let $(\hat{\alpha}_{ij})_{1 \leqslant i, j \leqslant k}$ be the second exterior power of the matrix (α_{ij}) (see [**1**]). Then $\wedge^2 F(G')$ is presented by

$$\left\langle A_1,\ldots,A_k \colon tA_i = \sum_{j=1}^{k} \hat{\alpha}_{ij} A_j,\ i = 1,\ldots,k \right\rangle. \tag{4}$$

Therefore, the presentation matrix $M(t)$ of this presentation is $tI - (\hat{\alpha}_{ij})$, where I is the $k \times k$ unit matrix. Since $\det(\alpha_{ij}) \neq 0$, it follows from [**1**, p. 640] that $\det M(0) = \pm\det(\hat{\alpha}_{ij}) \neq 0$. Therefore, by [**2**, Theorem 1.2], we obtain $r(\wedge^2 F(G')) = k$.

LEMMA 3.6. *If $r(F(G')) = 2$ or 3, then $H_2(F(G'))$ is not finitely generated over Z.*

PROOF. By [**1**, p. 640], we see that $\det(\hat{\alpha}_{ij}) = (\det(\alpha_{ij}))^{n-1}$ for $n = 2, 3$. Thus we have $\det M(0) \neq \pm 1, 0$. Hence, by [**15**], $\wedge^2 F(G')$ is not finitely generated over Z.

LEMMA 3.7. *Let $U = \langle a \colon t^{-1}a = 2a \rangle$ and V be a finite Λ-module. If V is a Λ-homomorphic image of U, then V is a cyclic group.*

PROOF. Let ρ be a Λ-epimorphism of U onto V, and μ the order of $\rho(a)$. Consider a quotient $W = U/\langle \mu a \rangle$. Then we have $W = \langle a \colon t^{-1}a = 2a, \mu a \rangle = \langle a \colon t^{-1}a = 2a, \mu' a \rangle$, where μ' is an integer such that $\mu = 2^i \mu'$ and $(\mu', 2) = 1$. Therefore W is cyclic of order μ'. Since V is a quotient of W, the proof is completed.

4. Proof of Theorem. Let n be the torsion-free rank of G'. By Lemmas 3.3 and 3.4, we have

$$r(G') = r\left(\overline{\tilde{F}(G')}\right) \geqslant r\left(H_2(G')\right) = r\left(H_2\left(F(G')\right)\right).$$

It follows from Lemma 3.5 that $n \geqslant n(n-1)/2$. Hence, we see that $1 \leqslant n \leqslant 3$.

Suppose that $n = 2$ or 3. Let $\tilde{E}_1$ and E_1 be the order ideals of $\overline{\tilde{F}(G')}$ and $\wedge^2 F(G')$, respectively. Then, from (3) and (4), as Λ-modules, $\overline{\tilde{F}(G')}$ and $\wedge^2 F(G')$ have square presentation matrices, and we see that $\tilde{E}_1 = (c_0 t^n + \cdots + c_{n-1}t + 1)$ and $E_1 = (t^k + d_{k-1}t^{k-1} + \cdots + d_0)$, where $k = n(n-1)/2$ and $c_i, d_i \in Z$, and $c_0, d_0 \neq 0$. Since $\wedge^2 F(G')$ is a Λ-homomorphic image of $\overline{\tilde{F}(G')}$, the polynomial $c_0 t^n + \cdots + 1$ must be contained in the ideal E_1. Hence, there exists a polynomial $h(t)$ of $Z[t]$ such that $c_0 t^n + \cdots + 1 = h(t)(t^k + \cdots + d_0)$. Therefore, d_0 must be ± 1. However, this contradicts Lemma 3.6. Thus we conclude that $r(G') = 1$. It follows from [5] that $G' \cong Z[1/2] \oplus T(G')$ and $\overline{\tilde{F}(G')} = \langle a \colon t^{-1}a = 2a \rangle$.

Next, we will consider the torsion part $T = T(G')$ of G'. For each prime p, let T_p be the p-primary component of T. Then, as an abelian group, we have

$$H_2(G') \cong \wedge^2 G' \cong \wedge^2 Z[1/2] \oplus \left(Z[1/2] \otimes T\right) \oplus \wedge^2 T \cong \bigoplus_{p \neq 2} T_p \oplus \wedge^2 T.$$

If the p-rank $r_p(T) \geqslant 3$, then we get $r_p(H_2(G')) \geqslant r_p(\wedge^2 T) \geqslant 3$. Furthermore, if $r_p(T) = 2$, then we have $p \neq 2$ from Proposition 3.2 and [5]. Hence, we obtain $r_p(H_2(G')) \geqslant r_p(T_p) = 2$. However, since $H_2(G')$ is a Λ-homomorphic image of $\overline{\tilde{F}(G')} = \langle a \colon t^{-1}a = 2a \rangle$, these contradict Lemma 3.7. Thus we conclude that $r_2(T) = 0$ and $r_p(T) \leqslant 1$ for any prime p $(\neq 2)$. (Note that $r_2(T) \neq 1$ [5].) Therefore, T is cyclic of odd order, and G' has a Λ-module presentation

$$\langle a, b \colon ta = 2a + \beta b,\ tb = \gamma b,\ \lambda b = 0 \rangle, \tag{5}$$

where λ is an odd integer, and $\beta, \gamma \in Z$. Moreover, using the arguments of pairing in [10], we see that γ must be -1.

Now, the second exterior power $\wedge^2 G'$ is a cyclic group of order λ generated by $a \wedge b$, and the action of t on $\wedge^2 G'$ is given by $t(a \wedge b) = -2(a \wedge b)$. Let ψ be the Λ-epimorphism of $\overline{\tilde{F}(G')}$ onto $\wedge^2 G'$ and let $\psi(a) = q(a \wedge b)$, $0 \leqslant q < \lambda$. Then, we have $t \cdot \psi(2a) = t(2q \cdot a \wedge b) = -4q(a \wedge b)$. On the other hand, we get $t \cdot \psi(2a) = \psi(t \cdot 2a) = \psi(a) = q(a \wedge b)$. Therefore, we obtain $5q(a \wedge b) = 0$, or $5q \equiv 0 \bmod \lambda$. Furthermore, since ψ is an epimorphism, it is clear that q is relatively prime to λ. Thus we have $5 \equiv 0 \bmod \lambda$, i.e., $\lambda = 1$ or 5. The proof is completed.

Remark. When $\lambda = 1$, from (5) G has a presentation

$$\langle a, t \colon t^{-1}at = a^2 \rangle.$$

Therefore, it can be realized as the group of a ribbon 2-knot in the 4-sphere S^4 [8]. In the case $\lambda = 5$, setting $a' = a + 2\beta b$ in (5), we have a Λ-module presentation for G'

$$\langle a', b \colon ta' = 2a',\ tb = -b,\ 5b = 0 \rangle.$$

Thus, we obtain (*) as a presentation for G. (It is easy to see that this group is an n-knot group, $n \geqslant 3$.)

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