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EXCESS LINEAR SERIES ON AN ALGEBRAIC CURVE[1]

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ABSTRACT. We prove that the dimensions of the varieties W_d^r of complete linear series of degree d and dimension at least r on a curve satisfy the inequalities $\dim W_{d-1}^r \geq \dim W_d^r - (r-1)$. In particular, a curve with ∞^2 g_d^1's must have a g_{d-1}^1.

In [4] it was seen that new results in the theory of linear systems on an algebraic curve could be obtained by using properties of ample vector bundles. The purpose of the present note is to make some further observations along the same lines. Specifically, for any curve C of genus g, we denote by $W_d^r = W_d^r(C)$ the variety of complete linear systems of degree d and dimension at least r on C. Our goal is to prove

THEOREM 1. *For any C,* $\dim W_{d-1}^r \geq \dim W_d^r - (r+1)$.

This theorem has a number of immediate consequences. For example, observing that, by Riemann-Roch,

$$W_d^r \cong W_{2g-2-d}^{g-d+r-1}, \qquad W_{d+1}^{r+1} \cong W_{2g-2-d-1}^{g-d+r-1},$$

and applying Theorem 1, we have

COROLLARY 2. $\dim W_{d+1}^{r+1} \geq \dim W_d^r - (g-d+r)$.

If we now combine Theorem 1 and Corollary 2, we have

COROLLARY 3. $\dim W_d^{r+1} \geq \dim W_d^r - (g-d+2r+2)$.

One special case of Theorem 1 is the statement that

$$\dim W_{d-1}^1 \geq \dim W_d^1 - 2.$$

In particular, if a curve C has a 2-dimensional family of g_d^1's, it must have a g_{d-1}^1.[2] This is in response to a question of Arthur Ogus.

REMARK. As will be apparent from the proof, Theorem 1 may be strengthened to say that if we choose *any* basepoint $p_0 \in C$ and correspondingly view W_{d-1}^r as a subvariety of W_d^r, and if Σ is any irreducible component of W_d^r, then

$$\dim(\Sigma \cap W_{d-1}^r) \geq \dim \Sigma - (r+1).$$

The other statements can be similarly sharpened.

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[2]The referee points out that this last statement has also been proved by M. Coppens (*Some sufficient conditions for the gonality of a smooth curve*, J. Pure Appl. Algebra **30** (1983), 5–21; Theorem 11).

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REMARK. On a general curve C of genus g, we have

$$\dim W^r_d = \rho = g - (r+1)(g-d+r)$$

and so equality holds in the theorem and corollaries when C is general. Thus these results go beyond the standard Brill-Noether theory exactly in case $\dim W^r_d > \rho$, that is, C has excess linear systems.

The proof of Theorem 1 is based on the easy

LEMMA 4. *Let E, F and F' be vector bundles of ranks m, n and $n+1$ on a projective variety X, π: $F' \to F$ a surjection with kernel L, σ: $E \to F'$ any bundle map, and Σ' (resp., Σ) the locus where σ (resp. $\pi \circ \sigma$) has rank k or less. Assume that $E^* \otimes L = \mathrm{Hom}(E, L)$ is ample. Then*

$$\dim \Sigma' \geq \dim \Sigma - (m-k).$$

PROOF. First, intersecting with a general subvariety H of codimension $\dim \Sigma - (m-k)$ in X, it is sufficient to show

$$\dim(\Sigma' \cap H) \geq \dim(\Sigma \cap H) - (m-k) = 0.$$

That is, we may assume $\dim \Sigma = m-k$, and then show that Σ' is not empty. Now if $\mathrm{rank}(\pi \circ \sigma)_x \leq k-1$ for any $x \in \Sigma$, then $\mathrm{rank}\, \sigma_x \leq k$ and we are done. We therefore assume $\mathrm{rank}(\pi \circ \sigma) = k$ throughout Σ, i.e. that $G = \mathrm{Ker}(\pi \circ \sigma)$ is a rank $m-k$ sub-vector-bundle of E. We then have an induced map $\tilde{\sigma}$: $G \to L$, and clearly Σ' is just the zero-locus in Σ of this map. But $\mathrm{Hom}(G, L) = G^* \otimes L$ is a quotient bundle of $E^* \otimes L$, and so, by hypothesis, ample. By the arguments of [**2, 4** or **1**, Chapter 7, §1], a global section of an ample vector bundle whose rank is less than or equal to the dimension of its base must have a zero.

To deduce Theorem 1 from the Lemma, we use the by-now standard determinantal description of W^r_d (cf. [**4**]): fix $e \geq 2g-d-1$ and denote by $\mathcal{L}$ the Poincaré line bundle on $C \times \mathrm{Pic}^e(C)$ (that is, the bundle whose restriction to each fiber $C \times \{L\}$ is isomorphic to L), normalized so that $\mathcal{L}|_{\{p_0\} \times \mathrm{Pic}^e(C)}$ is trivial; take $E = (\pi_2)_* \mathcal{L}$ (that is, the bundle whose fiber over $L \in \mathrm{Pic}^e(C)$ is $H^0(C, L)$) and, for distinct points $p_0, \ldots, p_{e-d} \in C$,

$$F = \bigoplus_{i=1}^{e-d} L_{p_i}, \qquad F' = \bigoplus_{i=0}^{e-d} L_{p_i},$$

where $L_p = \mathcal{L}|_{\{p\} \times \mathrm{Pic}^e(C)}$. Then W^r_d (resp., W^r_{d-1}) is, up to translation, the locus where the natural evaluation map $E \to F$ (resp., $E \to F'$) has rank $e-g-r$ or less. Since, finally, the bundle E^* is ample (cf. [**1** or **4**, §2]) we may apply Lemma 4 to conclude Theorem 1.

REMARK. Given a map ϕ from a bundle E to a bundle F with filtration $0 \subset F_1 \subset \cdots \subset F_k = F$ such that $E^* \otimes F_i/F_{i-1}$ is ample for each i, we may ask whether the codimensions of the various degeneracy loci associated to ϕ (as in [**3**]) in one another are necessarily no greater than expected. We do not know the answer to this, except that it is not always the case. For example, in the setting of Lemma 4 above, if Σ'' is the locus where $\pi \circ \sigma$ has rank $k-1$ or less, it is *not* necessarily true that

$$\dim \Sigma'' \geq \dim \Sigma' - (n-k+1),$$

although this does follow in the special case of the bundle map above by Serre duality, as in Corollary 2.

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