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ON THE NILPOTENCY INDEX OF THE RADICAL OF A GROUP ALGEBRA. VIII

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ABSTRACT. A characterization of a finite p-solvable group G with $t(G) < (r+2)(p-1)+1$ is given under certain conditions, where $t(G)$ is the nilpotency index of the radical of a group algebra of a finite p-solvable group G with a p-Sylow subgroup of order p^r over a field of characteristic p.

Let p be a fixed prime number, let G be a finite p-solvable group with a p-Sylow subgroup P of order p^r $(r \geq 1)$ and let $t(G)$ be the nilpotency index of the radical of a group algebra of G over a field of characteristic p.

The purpose of this paper is to prove the following lemma which contains essentially a result of our previous paper [**9**] (see also [**10**]).

LEMMA. *Assume that G has p-length at least 2 and P is regular. If $t(G) < (r+2)(p-1)+1$, then p is a Fermat prime and a 2-Sylow subgroup of $G/O_{p'}(G)$ is nonabelian.*

If G has p-length 1, then $t(G) = t(P)$ by [**3**, Theorem 2 and **11**, Theorem 1]. Hence as an easy consequence of this lemma together with [**5**, Theorem 6 and **4**, Theorem 1], we have the next corollary which contains [**5**, Theorem 6 and **10**, Theorem 1].

COROLLARY. *Assume that P is regular. If p is not a Fermat prime or a 2-Sylow subgroup of $G/O_{p'}(G)$ is abelian, then the following are equivalent.*

(1) $t(G) = r(p-1)+1$ *or* $(r+1)(p-1)+1$.

(2) $t(G) < (r+2)(p-1)+1$.

(3) *G has p-length 1 and P has a central element c of order p such that $P/<c>$ is elementary abelian.*

In particular, $t(G) = r(p-1)+1$ if and only if P is elementary abelian.

PROOF OF LEMMA. First we note that P is nonabelian by [**1**, Theorem 6.3.3, p. 228] and so p is odd by [**2**, Satz 3.10.3 a), p. 322]. We may assume $O_{p'}(G) = 1$ by $t(G) \geq t(G/O_{p'}(G))$ (see [**12**]). We set $U = O_p(G)$ and $|U| = p^s$. Inequalities $t(G) \geq t(G/U) + t(U) - 1$ and $t(G/U) \geq (r-s)(p-1)+1$ (see [**12**]) imply $t(U) < (s+2)(p-1)+1$. Thus we can see that U is of exponent p by [**5**, Theorem 6]. Since P is regular and $\langle x, U\rangle' \subseteq U$ for $x \in P$, it follows that $(xu)^p = x^p$ for $x \in P$ and $u \in U$. Hence we have

$$u^{x^{p-1}+\cdots+x+1} = u^{x^{p-1}} \cdots u^x u = x^{-p}(xu)^p = 1$$

for all $x \in P$ and $u \in U$, where $u^{x^s} = x^{-s}ux^s$ and $x^{p-1}+\cdots+x+1$ is the sum of endomorphisms $x^{p-1}, \ldots, x, 1$ of U. Let F be the Frattini subgroup of U. Noting

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that G/U is a subgroup of $GL(U/F)$ in view of [**1**, Theorem 6.3.4, p. 229], we can see that Hall-Higman's theorem [**1**, Theorem 11.1.1, p. 359] together with the last equation yields our assertion.

Next we shall present two examples relating our results.

There exist groups G satisfying the following conditions:

(1) G has p-length 2.

(2) P is not regular.

(3) A 2-Sylow subgroup of $G/O_{p'}(G)$ is abelian for $p \neq 2$.

(4) $t(G) < (r+2)(p-1)+1$.

(See [**7**, Proposition 3 and Lemma 5 and **6**, Examples 1, 2].)

The group $G = Qd(3)$ is an example such that

1. G has 3-length 2 and $O_{3'}(G) = 1$.
2. $P = M(3)$ is regular.
3. A 2-Sylow subgroup of G is a quaternion group of order 8.
4. $t(G) = 9 = (3+1)(3-1)+1$ for $p = 3$ (a Fermat prime !) (see [**8**]).

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