ABSTRACT. A characterization of a finite $p$-solvable group $G$ with $t(G) < (r + 2)(p - 1) + 1$ is given under certain conditions, where $t(G)$ is the nilpotency index of the radical of a group algebra of a finite $p$-solvable group $G$ with a $p$-Sylow subgroup of order $p^r$ over a field of characteristic $p$.

Let $p$ be a fixed prime number, let $G$ be a finite $p$-solvable group with a $p$-Sylow subgroup $P$ of order $p^r$ ($r > 1$) and let $t(G)$ be the nilpotency index of the radical of a group algebra of $G$ over a field of characteristic $p$.

The purpose of this paper is to prove the following lemma which contains essentially a result of our previous paper [9] (see also [10]).

**Lemma.** Assume that $G$ has $p$-length at least 2 and $P$ is regular. If $t(G) < (r + 2)(p - 1) + 1$, then $p$ is a Fermat prime and a 2-Sylow subgroup of $G/O_{p'}(G)$ is nonabelian.

If $G$ has $p$-length 1, then $t(G) = t(P)$ by [3, Theorem 2 and 11, Theorem 1]. Hence as an easy consequence of this lemma together with [5, Theorem 6 and 4, Theorem 1], we have the next corollary which contains [5, Theorem 6 and 10, Theorem 1].

**Corollary.** Assume that $P$ is regular. If $p$ is not a Fermat prime or a 2-Sylow subgroup of $G/O_{p'}(G)$ is abelian, then the following are equivalent.
1. $t(G) = r(p - 1) + 1$ or $(r + 1)(p - 1) + 1$.
2. $t(G) < (r + 2)(p - 1) + 1$.
3. $G$ has $p$-length 1 and $P$ has a central element $c$ of order $p$ such that $P/ (< c >$ is elementary abelian.

In particular, $t(G) = r(p - 1) + 1$ if and only if $P$ is elementary abelian.

**Proof of Lemma.** First we note that $P$ is nonabelian by [1, Theorem 6.3.3, p. 228] and so $p$ is odd by [2, Satz 3.10.3 a), p. 322]. We may assume $O_{p'}(G) = 1$ by $t(G) \geq (G/O_{p'}(G))$ (see [12]). We set $U = O_p(G)$ and $|U| = p^s$. Inequalities $t(G) \geq t(G/U) + t(U) - 1$ and $|U| = (r - s)(p - 1) + 1$ (see [12]) imply $t(U) < (s + 2)(p - 1) + 1$. Thus we can see that $U$ is of exponent $p$ by [5, Theorem 6]. Since $P$ is regular and $(x,U)/ \subseteq U$ for $x \in P$, it follows that $(xU)^p = xp^p$ for $x \in P$ and $u \in U$. Hence we have

$$u(x^{p-1} + \cdots + 1) = u^{x-1} \cdots u^x u = x^{-p}(xu)^p = 1$$

for all $x \in P$ and $u \in U$, where $u^x = x^{-s}ux^s$ and $x^{p-1} + \cdots + x + 1$ is the sum of endomorphisms $x^{p-1}, \ldots, 1$ of $U$. Let $F$ be the Frattini subgroup of $U$. Noting
that \( G/U \) is a subgroup of \( GL(U/F) \) in view of [1, Theorem 6.3.4, p. 229], we can see that Hall-Higman's theorem [1, Theorem 11.1.1, p. 359] together with the last equation yields our assertion.

Next we shall present two examples relating our results. There exist groups \( G \) satisfying the following conditions:

1. \( G \) has \( p \)-length 2.
2. \( P \) is not regular.
3. A 2-Sylow subgroup of \( G/O_{p'}(G) \) is abelian for \( p \neq 2 \).
4. \( t(G) < (r + 2)(p - 1) + 1 \).

(See [7, Proposition 3 and Lemma 5 and 6, Examples 1, 2].)

The group \( G = Qd(3) \) is an example such that

1. \( G \) has 3-length 2 and \( O_{3'}(G) = 1 \).
2. \( P = M(3) \) is regular.
3. A 2-Sylow subgroup of \( G \) is a quaternion group of order 8.
4. \( t(G) = 9 = (3 + 1)(3 - 1) + 1 \) for \( p = 3 \) (a Fermat prime!) (see [8]).

REFERENCES


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