PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 92, Number 3, November 1984

A NEW PROOF OF THE RAN-MATSUSAKA CRITERION FOR JACOBIANS

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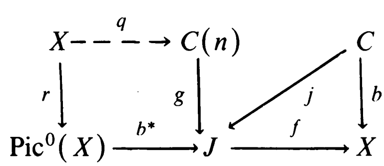
ABSTRACT. We give a characteristic free proof of a criterion which was first proved by Ran over **C**.

THEOREM. *Let X be an abelian variety of dimension n over an algebraically closed field. Let G be an effective one-cycle which generates X and let D be an ample divisor on X such that* $\deg(D \cdot G) = n$. *Then* (X, D, G) *is a Jacobian triple.*

REMARK. Ran's criterion [**3**] is an extension of Matsusaka's criterion [**1**]. Ran points out that Matsusaka's main tool is a certain endomorphism α; the composite map $f\,b^*r$ in the diagram below is the endomorphism $\alpha(G, D)$ of [**1**].

We consider first the case where G is reduced and irreducible. Let a: $C \to G$ be the normalization map, b: $C \to G \to X$ be the composite map, $C(n)$ denote the nth symmetric product of C, J be the jacobian of C and D_x be the translate of D by $x \in X$, i.e. $D_x = \{(d + x): d \in D\}$. Our main tool is the diagram



where $r(x) = \text{class}(D_x - D)$, $g(p_1, p_2, \ldots, p_n) = \text{class}(p_1 + p_2 + \cdots + p_n) - b^* \text{class}\, D$ and q is defined on the open set $V = \{x: G \not\subset D_x\}$ by the rule $q(x) = b^* D_x$, as a divisor on C. The map j is just the usual Abel-Jacobi morphism and f is induced by the universal property of $J = \text{Alb}(C)$.

Since D is ample, r is an isogeny. Ker b^* is finite because f is surjective and b^* is the dual map to f. Then $n = \dim X = \dim \text{closure}(gqV)$, so that qV is dense in $C(n)$, both having the same dimension. We have $b^*rX = gC(n)$, therefore $b^*rX = J$, because $gC(n)$ generates J and b^*r is a homomorphism. It follows that (i) $n = $ genus C, and (ii) f is an isogeny.

The divisor $E = f^*D$ is ample on J. Further we have

LEMMA. *E induces a principal polarization on J; hence* $\deg(E^n) = n!$.

Received by the editors May 20, 1983 and, in revised form, November 1, 1983.
1980 *Mathematics Subject Classification*. Primary 14K05, 14H40.
[1]Supported in part by a grant from the Italian Ministry of Public Education and the C.N.R.

PROOF. By [2, III, 16] it suffices to show $\dim|E| = 0$, i.e. that E is linearly isolated. If E is not linearly isolated, neither are its translates. Let $F = E_y$ be a general translate and $H \neq F$ a divisor which is linearly equivalent to F. It is easy to see that $C \cdot F$ and $C \cdot H$ are different divisors on C; hence $C \cdot F$ is a special divisor of degree n. Now $F = E_y = f^*D_x$, $x = f(y)$ and $C \cdot F = b^*D_x = q(x)$; therefore, for general x, $q(x)$ is a special divisor. This is a contradiction because qV is dense in $C(n)$.

PROPOSITION. $f\colon J \to X$ is an isomorphism.

PROOF. $n! = \deg(E^n) = \deg(f^*D)^n = \deg(f)\deg(D^n)$ and then $\deg(f) = 1$, because $\deg(D^n)/n! = \chi(O(D))$.

We identify J and X via f; hence $G = C$.

To finish we show that $-D = \{-d\colon d \in D\}$ is a translate of the standard image U of $C(n-1)$ in J, via the Abel-Jacobi map.

Let $\alpha\colon C(n) \to J$ be $\alpha = r^{-1}(b^*)^{-1}g$, α is well defined because r is an isomorphism, since D induces a principal polarization. The map q is the inverse of α; hence for the general point $(p_1,\ldots,p_n)$ of $C(n)$ we have $\alpha(p_1,\ldots,p_n) = x$, with $D_x \cdot C = p_1 + \cdots + p_n$.

Let $\beta\colon C \times C(n-1) \to J$ be given by $\beta(y; p_2,\ldots,p_n) = y - \alpha(y, p_2,\ldots,p_n)$, where y denotes both the point on C and its image in J.

LEMMA. $D = \operatorname{Image}\beta$.

PROOF. If $(y, p_2,\ldots,p_n)$ is a general point in $C(n)$, then $\alpha(y, p_2,\ldots,p_n) = x$, with $D_x \cdot C = y + p_2 + \cdots + p_n$ and hence $y \in D_x$, i.e. $(y - x) \in D$. It follows $\operatorname{Image}\beta \subseteq D$. Let $D'' = \operatorname{closure}(D - \operatorname{Image}\beta)$, we show $D'' = \varnothing$. If D'' is not empty it is a divisor in J. A general translate of C intersects D'' and $\operatorname{Image}\beta$ properly and outside of their intersection. Without restriction we may assume this translate to be C itself. Let $C \cdot D = w + q_2 + \cdots + q_n$ with $w \in D''$. Then $\beta(w; q_2,\ldots,q_n) = w$, a contradiction.

Up to a translation of D we may assume that $0 \in \operatorname{Image}\beta$. By the rigidity lemma [2, II, 4] there are morphisms $\gamma\colon C \to J$ and $\delta\colon C(n-1) \to J$ such that $\beta(y; p_2,\ldots,p_n) = \gamma(y) + \delta(p_2,\ldots,p_n)$. Fixing y we notice that

$$\dim \beta(\{y\} \times C(n-1)) = (n-1),$$

because α is generically injective and hence $\beta(\{y\} \times C(n-1)) = D$ for all y. It follows that $\gamma(C) = \{0\}$, because D is not left fixed by any translation; hence

$$\delta(p_2,\ldots,p_n) = \beta(y; p_2,\ldots,p_n) = y - \alpha(y, p_2,\ldots,p_n) \quad \forall y \in C.$$

Fix now $n-1$ points, $z_1,\ldots,z_{n-1}$, on C. For any point $(p_1; p_2,\ldots,p_n)$ of $C \times C(n-1)$ one has

$$\begin{aligned}\beta(p_1; p_2,\ldots,p_n) &= \delta(p_2,\ldots,p_n) = -\alpha(z_1, p_2,\ldots,p_n) + z_1\\ &= -p_2 + \delta(p_3,\ldots,p_n, z_1) + z_1 = -p_2 - p_3 + \delta(p_4,\ldots,p_n, z_1, z_2) + z_1 + z_2\\ &= \cdots = -(p_2 + \cdots + p_n) + (\delta(z_1, z_2,\ldots,z_{n-1}) + z_1 + z_2 + \cdots + z_{n-1}).\end{aligned}$$

Since $D = \beta(C \times C(n-1))$, D is a translate of $-U$, where $U = \{(y_1 + \cdots + y_{n-1})\colon y_i \in C\}$.

Next one should deal with the case when G is not reduced and irreducible. In a first draft of this paper we sketched the proof, which follows exactly the same pattern as in the preceding case. Following the referee's advice we omit it because it is a rather straightforward exercise, cf. p. 469 in [**3**].

REMARK. The theorem is still true if the apparently weaker hypothesis $\deg(D \cdot G) \leqslant n$ was given instead of the equality. The proof is the same.

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