A NEW PROOF OF THE RAN-MATSUSAKA CRITERION
FOR JACOBIANS

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Abstract. We give a characteristic free proof of a criterion which was first proved by Ran over $\mathbb{C}$.

Theorem. Let $X$ be an abelian variety of dimension $n$ over an algebraically closed field. Let $G$ be an effective one-cycle which generates $X$ and let $D$ be an ample divisor on $X$ such that $\deg(D \cdot G) = n$. Then $(X, D, G)$ is a Jacobian triple.

Remark. Ran's criterion [3] is an extension of Matsusaka's criterion [1]. Ran points out that Matsusaka's main tool is a certain endomorphism $\alpha$; the composite map $f \circ b^* r$ in the diagram below is the endomorphism $\alpha(G, D)$ of [1].

We consider first the case where $G$ is reduced and irreducible. Let $a: C \to G$ be the normalization map, $b: C \to G \to X$ be the composite map, $C(n)$ denote the $n$th symmetric product of $C$, $J$ be the jacobian of $C$ and $D_x$ be the translate of $D$ by $x \in X$, i.e. $D_x = \{(d + x): d \in D\}$. Our main tool is the diagram

$$
\begin{array}{cccc}
X & \overset{q}{\longrightarrow} & C(n) & \overset{C}{\longrightarrow} \\
\downarrow{r} & & \downarrow{g} & \\
Pic^0(X) & \overset{b^*}{\longrightarrow} & J & \overset{f}{\longrightarrow} X
\end{array}
$$

where $r(x) = \text{class}(D_x - D)$, $g(p_1, p_2, \ldots, p_n) = \text{class}(p_1 + p_2 + \cdots + p_n) - b^* \text{class } D$ and $q$ is defined on the open set $V = \{x: G \notin D_x\}$ by the rule $q(x) = b^*D_x$, as a divisor on $C$. The map $j$ is just the usual Abel-Jacobi morphism and $f$ is induced by the universal property of $J = \text{Alb}(C)$.

Since $D$ is ample, $r$ is an isogeny. $\text{Ker } b^*$ is finite because $f$ is surjective and $b^*$ is the dual map to $f$. Then $n = \dim X = \dim \text{closure}(gqV)$, so that $qV$ is dense in $C(n)$, both having the same dimension. We have $b^* r X = gC(n)$, therefore $b^* r X = J$, because $gC(n)$ generates $J$ and $b^* r$ is a homomorphism. It follows that (i) $n = \text{genus } C$, and (ii) $f$ is an isogeny.

The divisor $E = f^* D$ is ample on $J$. Further we have

Lemma. $E$ induces a principal polarization on $J$; hence $\deg(E^n) = n!$.

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Proof. By [2, III, 16] it suffices to show \(\dim|E| = 0\), i.e. that \(E\) is linearly isolated. If \(E\) is not linearly isolated, neither are its translates. Let \(F = E_x\) be a general translate and \(H \neq F\) a divisor which is linearly equivalent to \(F\). It is easy to see that \(C \cdot F\) and \(C \cdot H\) are different divisors on \(C\); hence \(C \cdot F\) is a special divisor of degree \(n\). Now \(F = E_y = f^*D_x\), \(x = f(y)\) and \(C \cdot F = b^*D_x = q(x)\); therefore, for general \(x\), \(q(x)\) is a special divisor. This is a contradiction because \(qV\) is dense in \(C(n)\).

Proposition. \(f: J \to X\) is an isomorphism.

Proof. \(n! = \deg(E^n) = \deg(f^*D)^n = \deg(f)\deg(D^n)\) and then \(\deg(f) = 1\), because \(\deg(D^n)/n! = \chi(O(D))\). We identify \(J\) and \(X\) via \(f\); hence \(G = C\).

To finish we show that \(-D = \{-d: d \in D\}\) is a translate of the standard image \(U\) of \(C(n - 1)\) in \(J\), via the Abel-Jacobi map.

Let \(\alpha: C(n) \to J\) be \(\alpha = r^{-1}(b*)^{-1}g\), \(\alpha\) is well defined because \(r\) is an isomorphism, since \(D\) induces a principal polarization. The map \(q\) is the inverse of \(\alpha\); hence for the general point \((p_1, \ldots, p_n)\) of \(C(n)\) we have \(\alpha(p_1, \ldots, p_n) = x\), with \(D_x \cdot C = p_1 + \cdots + p_n\).

Let \(\beta: C \times C(n - 1) \to J\) be given by \(\beta(y; p_2, \ldots, p_n) = y - \alpha(y, p_2, \ldots, p_n)\), where \(y\) denotes both the point on \(C\) and its image in \(J\).

Lemma. \(D = \Image \beta\).

Proof. If \((y, p_2, \ldots, p_n)\) is a general point in \(C(n)\), then \(\alpha(y, p_2, \ldots, p_n) = x\), with \(D_x \cdot C = y + p_2 + \cdots + p_n\) and hence \(y \in D_x\), i.e. \((y - x) \in D\). It follows \(\Image \beta \subseteq D\). Let \(D''\) = closure \((D - \Image \beta)\), we show \(D'' = \emptyset\). If \(D''\) is not empty it is a divisor in \(J\). A general translate of \(C\) intersects \(D''\) and \(\Image \beta\) properly and outside of their intersection. Without restriction we may assume this translate to be \(C\) itself. Let \(C \cdot D = w + q_2 + \cdots + q_n\) with \(w \in D''\). Then \(\beta(w; q_2, \ldots, q_n) = w\), a contradiction.

Up to a translation of \(D\) we may assume that \(0 \in \Image \beta\). By the rigidity lemma [2, II, 4] there are morphisms \(\gamma: C \to J\) and \(\delta: C(n - 1) \to J\) such that \(\beta(y; p_2, \ldots, p_n) = \gamma(y) + \delta(p_2, \ldots, p_n)\). Fixing \(y\) we notice that

\[
\dim \beta(\{y\} \times C(n - 1)) = (n - 1),
\]

because \(\alpha\) is generically injective and hence \(\beta(\{y\} \times C(n - 1)) = D\) for all \(y\). It follows that \(\gamma(C) = \{0\}\), because \(D\) is not left fixed by any translation; hence

\[
\delta(p_2, \ldots, p_n) = \beta(y; p_2, \ldots, p_n) = y - \alpha(y, p_2, \ldots, p_n) \quad \forall y \in C.
\]

Fix now \(n - 1\) points, \(z_1, \ldots, z_{n-1}\), on \(C\). For any point \((p_1, p_2, \ldots, p_n)\) of \(C \times C(n - 1)\) one has

\[
\beta(p_1; p_2, \ldots, p_n) = \delta(p_2, \ldots, p_n) = -\alpha(z_1, p_2, \ldots, p_n) + z_1
\]

\[
= -p_2 + \delta(p_3, \ldots, p_n, z_1) + z_1 = -p_2 - p_3 + \delta(p_4, \ldots, p_n, z_1, z_2) + z_1 + z_2
\]

\[
= \cdots = -(p_2 + \cdots + p_n) + \delta(z_1, z_2, \ldots, z_{n-1}) + z_1 + z_2 + \cdots + z_{n-1}.
\]

Since \(D = \beta(C \times C(n - 1))\), \(D\) is a translate of \(-U\), where \(U = \{(y_1 + \cdots + y_{n-1}): y_i \in C\}\).
Next one should deal with the case when $G$ is not reduced and irreducible. In a first draft of this paper we sketched the proof, which follows exactly the same pattern as in the preceding case. Following the referee's advice we omit it because it is a rather straightforward exercise, cf. p. 469 in [3].

Remark. The theorem is still true if the apparently weaker hypothesis $\deg(D \cdot G) \leq n$ was given instead of the equality. The proof is the same.

References


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