

## A NEW PROOF OF THE RAN-MATSUSAKA CRITERION FOR JACOBIANS

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ABSTRACT. We give a characteristic free proof of a criterion which was first proved by Ran over  $\mathbb{C}$ .

**THEOREM.** *Let  $X$  be an abelian variety of dimension  $n$  over an algebraically closed field. Let  $G$  be an effective one-cycle which generates  $X$  and let  $D$  be an ample divisor on  $X$  such that  $\deg(D \cdot G) = n$ . Then  $(X, D, G)$  is a Jacobian triple.*

**REMARK.** Ran's criterion [3] is an extension of Matsusaka's criterion [1]. Ran points out that Matsusaka's main tool is a certain endomorphism  $\alpha$ ; the composite map  $f b^* r$  in the diagram below is the endomorphism  $\alpha(G, D)$  of [1].

We consider first the case where  $G$  is reduced and irreducible. Let  $a: C \rightarrow G$  be the normalization map,  $b: C \rightarrow G \rightarrow X$  be the composite map,  $C(n)$  denote the  $n$ th symmetric product of  $C$ ,  $J$  be the jacobian of  $C$  and  $D_x$  be the translate of  $D$  by  $x \in X$ , i.e.  $D_x = \{(d + x): d \in D\}$ . Our main tool is the diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{q} & C(n) & & C \\
 r \downarrow & & g \downarrow & \swarrow j & \downarrow b \\
 \text{Pic}^0(X) & \xrightarrow{b^*} & J & \xrightarrow{f} & X
 \end{array}$$

where  $r(x) = \text{class}(D_x - D)$ ,  $g(p_1, p_2, \dots, p_n) = \text{class}(p_1 + p_2 + \dots + p_n) - b^* \text{class } D$  and  $q$  is defined on the open set  $V = \{x: G \not\subset D_x\}$  by the rule  $q(x) = b^* D_x$ , as a divisor on  $C$ . The map  $j$  is just the usual Abel-Jacobi morphism and  $f$  is induced by the universal property of  $J = \text{Alb}(C)$ .

Since  $D$  is ample,  $r$  is an isogeny.  $\text{Ker } b^*$  is finite because  $f$  is surjective and  $b^*$  is the dual map to  $f$ . Then  $n = \dim X = \dim \text{closure}(gqV)$ , so that  $qV$  is dense in  $C(n)$ , both having the same dimension. We have  $b^* r X = gC(n)$ , therefore  $b^* r X = J$ , because  $gC(n)$  generates  $J$  and  $b^* r$  is a homomorphism. It follows that (i)  $n = \text{genus } C$ , and (ii)  $f$  is an isogeny.

The divisor  $E = f^* D$  is ample on  $J$ . Further we have

**LEMMA.**  *$E$  induces a principal polarization on  $J$ ; hence  $\deg(E^n) = n!$ .*

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PROOF. By [2, III, 16] it suffices to show  $\dim|E| = 0$ , i.e. that  $E$  is linearly isolated. If  $E$  is not linearly isolated, neither are its translates. Let  $F = E_y$  be a general translate and  $H \neq F$  a divisor which is linearly equivalent to  $F$ . It is easy to see that  $C \cdot F$  and  $C \cdot H$  are different divisors on  $C$ ; hence  $C \cdot F$  is a special divisor of degree  $n$ . Now  $F = E_y = f^*D_x$ ,  $x = f(y)$  and  $C \cdot F = b^*D_x = q(x)$ ; therefore, for general  $x$ ,  $q(x)$  is a special divisor. This is a contradiction because  $qV$  is dense in  $C(n)$ .

PROPOSITION.  $f: J \rightarrow X$  is an isomorphism.

PROOF.  $n! = \deg(E^n) = \deg(f^*D)^n = \deg(f)\deg(D^n)$  and then  $\deg(f) = 1$ , because  $\deg(D^n)/n! = \chi(O(D))$ .

We identify  $J$  and  $X$  via  $f$ ; hence  $G = C$ .

To finish we show that  $-D = \{-d: d \in D\}$  is a translate of the standard image  $U$  of  $C(n-1)$  in  $J$ , via the Abel-Jacobi map.

Let  $\alpha: C(n) \rightarrow J$  be  $\alpha = r^{-1}(b^*)^{-1}g$ ,  $\alpha$  is well defined because  $r$  is an isomorphism, since  $D$  induces a principal polarization. The map  $q$  is the inverse of  $\alpha$ ; hence for the general point  $(p_1, \dots, p_n)$  of  $C(n)$  we have  $\alpha(p_1, \dots, p_n) = x$ , with  $D_x \cdot C = p_1 + \dots + p_n$ .

Let  $\beta: C \times C(n-1) \rightarrow J$  be given by  $\beta(y; p_2, \dots, p_n) = y - \alpha(y, p_2, \dots, p_n)$ , where  $y$  denotes both the point on  $C$  and its image in  $J$ .

LEMMA.  $D = \text{Image } \beta$ .

PROOF. If  $(y, p_2, \dots, p_n)$  is a general point in  $C(n)$ , then  $\alpha(y, p_2, \dots, p_n) = x$ , with  $D_x \cdot C = y + p_2 + \dots + p_n$  and hence  $y \in D_x$ , i.e.  $(y - x) \in D$ . It follows  $\text{Image } \beta \subseteq D$ . Let  $D'' = \text{closure}(D - \text{Image } \beta)$ , we show  $D'' = \emptyset$ . If  $D''$  is not empty it is a divisor in  $J$ . A general translate of  $C$  intersects  $D''$  and  $\text{Image } \beta$  properly and outside of their intersection. Without restriction we may assume this translate to be  $C$  itself. Let  $C \cdot D = w + q_2 + \dots + q_n$  with  $w \in D''$ . Then  $\beta(w; q_2, \dots, q_n) = w$ , a contradiction.

Up to a translation of  $D$  we may assume that  $0 \in \text{Image } \beta$ . By the rigidity lemma [2, II, 4] there are morphisms  $\gamma: C \rightarrow J$  and  $\delta: C(n-1) \rightarrow J$  such that  $\beta(y; p_2, \dots, p_n) = \gamma(y) + \delta(p_2, \dots, p_n)$ . Fixing  $y$  we notice that

$$\dim \beta(\{y\} \times C(n-1)) = (n-1),$$

because  $\alpha$  is generically injective and hence  $\beta(\{y\} \times C(n-1)) = D$  for all  $y$ . It follows that  $\gamma(C) = \{0\}$ , because  $D$  is not left fixed by any translation; hence

$$\delta(p_2, \dots, p_n) = \beta(y; p_2, \dots, p_n) = y - \alpha(y, p_2, \dots, p_n) \quad \forall y \in C.$$

Fix now  $n-1$  points,  $z_1, \dots, z_{n-1}$ , on  $C$ . For any point  $(p_1; p_2, \dots, p_n)$  of  $C \times C(n-1)$  one has

$$\begin{aligned} \beta(p_1; p_2, \dots, p_n) &= \delta(p_2, \dots, p_n) = -\alpha(z_1, p_2, \dots, p_n) + z_1 \\ &= -p_2 + \delta(p_3, \dots, p_n, z_1) + z_1 = -p_2 - p_3 + \delta(p_4, \dots, p_n, z_1, z_2) + z_1 + z_2 \\ &= \dots = -(p_2 + \dots + p_n) + (\delta(z_1, z_2, \dots, z_{n-1}) + z_1 + z_2 + \dots + z_{n-1}). \end{aligned}$$

Since  $D = \beta(C \times C(n-1))$ ,  $D$  is a translate of  $-U$ , where  $U = \{(y_1 + \dots + y_{n-1}): y_i \in C\}$ .

Next one should deal with the case when  $G$  is not reduced and irreducible. In a first draft of this paper we sketched the proof, which follows exactly the same pattern as in the preceding case. Following the referee's advice we omit it because it is a rather straightforward exercise, cf. p. 469 in [3].

REMARK. The theorem is still true if the apparently weaker hypothesis  $\deg(D \cdot G) \leq n$  was given instead of the equality. The proof is the same.

#### REFERENCES

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