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PROPERTIES OF THE FOURIER ALGEBRA THAT ARE EQUIVALENT TO AMENABILITY

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ABSTRACT. It is shown that a locally compact group G is amenable iff each multiplier on the Fourier algebra $A(G)$ is given by a function from the Fourier-Stieltjes algebra $B(G)$. Another condition is that the norm of $A(G)$ is equivalent to that induced by the regular representation of $A(G)$.

Several properties of $A(G)$ have been shown to be equivalent to the amenability of G: (1) $A(G)$ has a bounded approximate identity [**Lep 2**; **H 1**, Theorem 6, p. 120]. (2) $A(G)$ factorizes weakly, i.e. $A(G)$ is the linear span of $A(G) \cdot A(G)$ [**Lo 1**, Proposition 2, p. 138]. (3) $M(A(G), A(G)) = B(G)$ (G discrete), where $M(\cdot)$ denotes the space of multipliers (see below). This was shown by Nebbia [**N**, Theorem 2, p. 553]. In the case of groups having nonabelian free subgroups, the existence of multipliers on $A(G)$ not belonging to $B(G)$ was shown by Figà-Talamanca and Picardello [**FP**]. Some results on the case of Lie groups can be found in [**DH**]. In this paper we extend (3) to nondiscrete groups.

THEOREM 1. *The following statements are equivalent for any locally compact group* G:

(a) G *is amenable.*

(b) $M(A(G), A(G)) = B(G)$.

(c) *The norm of* $A(G)$ *is equivalent to that induced by the regular representation of* $A(G)$ (*by multiplication on itself*).

Condition (c) means there exists $c > 0$ such that, for each $u \in A(G)$,

$$\sup\{\|uv\|_A \colon v \in A(G), \ \|v\|_A \leq 1\} \geq c\|u\|_A,$$

or, equivalently (by the open mapping theorem), $A(G)$ is closed in $M(A(G), A(G))$. In the case of amenable groups, (c) (and the factorization property of $A(G)$) are consequences of the existence of a bounded approximate identity (and one even gets an isometry, i.e. $c = 1$ in (c) and ordinary factorization). But for general Banach algebras, the reverse conclusion is not possible, as was shown by an example of Leinert [**Lei**].

Notations. e denotes the unit of G, λ a fixed Haar measure on G. In integrals, dx, dy, etc. refer to λ. For $f \colon G \to \mathbf{C}$, $x \in G$, we define *left translation* by $L_x f(y) = f(x^{-1}y)$. Similarly, for $g \in L^1(G)$, $f \in L^p(G)$, we use convolution to define $L_g f = g * f$. For $x \in G$, δ_x denotes the Dirac measure concentrated at x. $\mathcal{K}(G)$ denotes the space of continuous (complex-valued) functions f on G with *compact support* $\operatorname{supp} f$. $C_{\text{lu}}(G)$ denotes the space of *left-uniformly continuous functions*; it

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consists of those bounded functions $f\colon G \to \mathbf{C}$ for which $x \to L_x f$ is continuous with respect to $\| \ \|_\infty$. For the definition and simple properties of means on $L^\infty(G)$ and $C_{\mathrm{lu}}(G)$, we refer to [**Gr**]. G is called *amenable* if there exists a left-invariant mean on $L^\infty(G)$.

For the definition and properties of the *Fourier algebra* $(A(G), \| \ \|_A)$ and the *Fourier-Stieltjes algebra* $B(G)$, we refer to [**E**]. $P_1(G)$ denotes the set of *continuous positive-definite functions* $u\colon G \to \mathbf{C}$ such that $u(e) = 1$. By [**E**, p. 218], any $u \in A(G)$ can be written as $u(x) = (L_x h, k)$, where $h, k \in L^2(G)$ and $(\ , \ )$ denotes the inner product of $L^2(G)$. If, in addition, $u \in P_1(G)$, one can assume that $h = k$ and $\|h\|_2 = 1$ [**E**, p. 188]. $\mathrm{VN}(G)$ denotes the *von Neumann algebra* on $L^2(G)$ generated by the operators L_x $(x \in G)$, $C_L^*(G)$ denotes the C^**-algebra* generated by all L_f $(f \in L^1(G))$, i.e. the norm closure in the space of operators. By [**E**, p. 210], $\mathrm{VN}(G)$ can be identified with the dual of $A(G)$, for $T \in \mathrm{VN}(G)$, $u \in A(G)$ as above, the duality is given by $\langle T, u \rangle = (Th, k)$.

A linear operator $\Gamma\colon A(G) \to A(G)$ is called an $(A(G)$-$)$*multiplier* if $\Gamma(uv) = u\Gamma(v)$ for all $u, v \in A(G)$ (similarly for arbitrary $A(G)$-modules). The space of multipliers is denoted by $M(A(G), A(G))$. It follows easily from [**E**, (3.34), p. 222] and the closed graph theorem that any $\Gamma \in M(A(G), A(G))$ is norm bounded and given by a multiplication operator $\Gamma(u) = gu$, where $g\colon G \to \mathbf{C}$ is bounded and continuous. Conversely, any $g \in B(G)$ defines a multiplier [**E**, (3.4), p. 208], i.e. $B(G) \subseteq M(A(G), A(G))$.

LEMMA 1. *If $T \in C_L^*(G)$ and K is a compact subset of G, then $T \mid L^2(K)$ is compact.* (*In particular, $C_L^*(G)$ has a unit iff G is discrete.*)

PROOF. $T \mid L^2(K)$ denotes the restriction of T considered as a mapping from $L^2(K)$ (subspace of $L^2(G)$) to $L^2(G)$. We may assume $T = L_f$, with $f \in \mathcal{K}(G)$, and $\lambda(K) > 0$. If $g \in L^2(K)$ with $\|g\|_2 \le \lambda(K)^{-1/2}$, then $\|g\|_1 \le 1$; hence $L_f(g) = f * g = \int_K g(x) f * \delta_x \, dx$ belongs to the closed, absolutely convex hull of $\{f * \delta_x\colon x \in K\}$. (If L_f is invertible, this is possible only if $L^2(K)$ is finite dimensional.)

REMARK. Even if G is not discrete, it may happen that the spectrum of $C_L^*(G)$ (or $C^*(G)$) is compact (see [**B**, p. 144]).

In Lemma 2 and Proposition 1, f denotes a continuous function on $\mathbf{R}$ such that $0 \le f \le 1$, $f(t) = 0$ for $t \le 1/4$ and $f(t) = 1$ for $t \ge 1/2$.

LEMMA 2. (a) *If $R \in \mathrm{VN}(G)$, $R \le 1$, $u \in A(G) \cap P_1(G)$, $x \in G$, $\langle R, u \rangle > 1 - \delta^2/2$, $\langle L_{x^{-1}} R L_x, u \rangle > 1 - \delta^2/2$, $R' = f(R)$, $S = R' L_x R'$, then $\langle S, L_x u \rangle > 1 - 2\delta$.*

(b) *If R, u are as in* (a), *$R_i \in \mathrm{VN}(G)$ commute pairwise, $0 \le R_i \le 1$ $(i = 1, \ldots, n)$, $\langle \sum_{i=1}^n R_i^2, u \rangle < (\delta^2/4n)^2$, and $R'' = \prod_{i=1}^n (1 - R_i) \, R \prod_{i=1}^n (1 - R_i)$, then $\langle R'', u \rangle > 1 - \delta^2$.*

PROOF. (a) We have $u(x) = (L_x h, h)$ for some $h \in L^2(G)$ with $\|h\|_2 = 1$. Since $(1 - R')^2 \le 2(1 - R)$, we get

$$\|h - R'h\|_2^2 \le 2((1 - R)h, h) = 2(1 - \langle R, u \rangle) < \delta^2$$

and, similarly, $\|L_x h - R' L_x h\|_2^2 < \delta^2$. Now

$$\langle S, L_x u \rangle = (Sh, L_x h) = (L_x R' h, R' L_x h) > (L_x h, L_x h) - 2\delta = 1 - 2\delta.$$

(b) Keeping the notation of (a), we have $\|R_i h\|_2 < \delta^2/4n$. Since

$$1 - \prod_{i=1}^{n}(1 - R_i) = \sum_{i=1}^{n} R_i \prod_{j=1}^{i-1}(1 - R_j),$$

we get $\| \prod_{i=1}^{n}(1 - R_i)h - h\|_2 < \delta^2/4$ and the result follows.

In the case of a nondiscrete abelian group G, multiplication by characters defines an isometry on $A(G)$ and, given $u_1, u_2 \in A(G)$, one can find $\chi_1, \chi_2 \in G\widehat{\ }$ such that $(u_1\chi_1)\widehat{\ }$ and $(u_2\chi_2)\widehat{\ }$ have "almost" disjoint supports. The following proposition shows that similar things can be done for general G, at least for translates of positive-definite functions.

PROPOSITION 1. *Assume that G is not discrete*, $u_i \in A(G) \cap P_1(G)$, $x_i \in G$ $(i = 1, \ldots, n)$, $\varepsilon > 0$. *Then there exist* $w_i \in P_1(G)$, $S_i \in C_L^*(G)$ *satisfying*: $\|S_i\| \leq 1$, *the images of* S_i *(resp.* S_i^**) are pairwise orthogonal,* $\langle S_i, L_{x_i}(u_i w_i)\rangle > 1 - \varepsilon$ *for all* i.

In particular,

$$\left\| \sum_{i=1}^{n} \mu_i L_{x_i}(u_i w_i) \right\|_A > (1 - \varepsilon) \sum_{i=1}^{n} |\mu_i| \quad \textit{for all } \mu_i \in \mathbf{C}.$$

PROOF. We use induction on n. Observe that $C_L^*(G)$ is w*-dense in VN(G) [**Di 1**, I.3.4, Corollary 1, p. 45] and that the same is true for the hermitian parts of the unit balls [**Di 1**, I.3.5, Theorem 3, p. 47]. For $n = 1$ take $R \in C_L^*(G)$ such that $R \leq 1$, $\langle R, u_1\rangle > 1 - \varepsilon^2/8$ and $\langle L_{x_1^{-1}} R L_{x_1}, u_1\rangle > 1 - \varepsilon^2/8$. Put $R_1 = f(R)$, $\overline{R}_1 = f(2R)$, $S_1 = R_1 L_{x_1} R_1$. Then by Lemma 2(a) $\langle S_1, L_{x_1} u_1\rangle > 1 - \varepsilon$. Put $w_1 = 1$.

Now assume we have constructed S_i and w_i $(i = 1, \ldots, n)$ as needed, so that $S_i = R_i L_{x_i} R_i$, $R_i, \overline{R}_i \in C_L^*(G)$, $0 \leq R_i, \overline{R}_i \leq 1$, $\overline{R}_i \overline{R}_j = 0$ for $i \neq j$ and $R_i \overline{R}_i = R_i$. Put $T_1 = \sum_{i=1}^{n} \overline{R}_i^2$, $T_2 = L_{x_{n+1}^{-1}} T_1 L_{x_{n+1}}$. Then $u_{n+1} T_k \in C_L^*(G)$ [**E**, p. 224]. By Lemma 1 there exists $h \in L^2(G)$ such that $\|h\|_2 = 1$ and $((u_{n+1} T_k)h, h) < (\varepsilon^2/64n)^2$ for $k = 1, 2$. Put $w_{n+1}(x) = (L_x h, h)$. Then $\langle T_k, u_{n+1} w_{n+1}\rangle < (\varepsilon^2/64n)^2$.

Now, as above, choose $R \in C_L^*(G)$ such that $R \leq 1$, $\langle R, u_{n+1} w_{n+1}\rangle > 1 - \varepsilon^2/16$, $\langle L_{x_{n+1}^{-1}} R L_{x_{n+1}}, u_{n+1} w_{n+1}\rangle > 1 - \varepsilon^2/16$. Put

$$R' = \prod_{i=1}^{n}(1 - \overline{R}_i)\, R \prod_{i=1}^{n}(1 - \overline{R}_i), \quad R_{n+1} = f(R'), \quad \overline{R}_{n+1} = f(2R').$$

Then $R' R_i = 0$ for $i = 1, \ldots, n$. Using Lemma 2(b) and 2(a) and replacing $\overline{R}_i$ by $(1 - f(4R'))\overline{R}_i(1 - f(4R'))$ for $i = 1, \ldots, n$, we get all properties needed for the induction step.

LEMMA 3. *Assume that G is not discrete and* $\sup\{\|uv\|_A \colon \|v\|_A \leq 1\} > c\|u\|_A$ *for all* $u \in A(G)$. *Then, given* $u \in A(G) \cap P_1(G)$, $x_1, \ldots, x_n \in G$, *there exists* $v \in A(G) \cap P_1(G)$ *such that* $\sum_{i=1}^{n} \|v \cdot L_{x_i} u\|_A > cn/4$.

PROOF. By Proposition 1 there exists $w_i \in P_1(G)$ such that $\| \sum_{i=1}^{n} L_{x_i}(u w_i)\|_A > n/2$. By assumption there exists $v \in A(G)$ with $\|v\|_A \leq 1$ such that

$$\left\| v \cdot \sum_{i=1}^{n} L_{x_i}(u w_i) \right\|_A > \frac{cn}{2}.$$

Replacing $cn/2$ by $cn/4$, we may assume that v is hermitian. Then by [**E**, (3.15), p. 212 and (2.7), p. 193] we get v positive-definite. Now the result follows.

The next step uses a device due to H. Rindler (see [**Lo 1**, Lemma 4, p. 136]).

LEMMA 4. *For $a, b \in L^2(G)$ with $a, b \geq 0$, we have*

$$\int_G |a^2 - b^2|\, d\lambda \leq \left(\left(\int_G a^2 + b^2\, d\lambda\right)^2 - 4\left(\int_G ab\, d\lambda\right)^2\right)^{1/2}.$$

LEMMA 5. *Let $0 < \varepsilon < 1$, U, V, W be e-neighbourhoods such that $UW \subseteq V$, $\lambda(V) < (1+\varepsilon)\lambda(W)$. Let $h, k \in L^2(G)$, $T \in \mathrm{VN}(G)$ be such that $\|h\|_2 \leq 1$, $\|k\|_2 \leq 1$, $\|T\| \leq 1, \operatorname{supp} T \subseteq U$. Define a mean M on $C_{\mathrm{lu}}(G)$ by $M(f) = \int_G f \cdot |h|^2\, d\lambda$. Then for $x \in G$, $f \in C_{\mathrm{lu}}(G)$,*

$$\begin{aligned}|M(L_x f - f)| \leq & 2\sup\{\|L_y f - f\|_\infty : y \in V\} \\ & + \|f\|_\infty (2 + 3\varepsilon - |(T(h \otimes k), L_{x^{-1}} h \otimes k)|^2).\end{aligned}$$

(See [**E**, 4.5, p. 226] for the definition of $\operatorname{supp} T$; the representation $x \to L_x \otimes L_x$ is quasi-equivalent to L [**D 2**, Example 13.11.3, p. 306], hence T is also defined on $L^2(G) \otimes L^2(G)$ [**D 2**, 5.3.1, p. 118].)

PROOF. Put $\delta = \sup\{\|L_y f - f\|_\infty : y \in V\}$. Since $\|k\|_2 = 1$, we have

$$M(f) = \int_{G\times G} f(y) |h(y) k(z)|^2 d(y, z).$$

$$\begin{aligned}\Bigg|\int_{G\times G} & f(y) |h \otimes k|^2(y, z)\, d(y, z) \\ & - \lambda(V)^{-1} \int_V \int_{G\times G} f(t^{-1}y) |h \otimes k|^2(y, z)\, d(y, z)\, dt\Bigg| < \delta\end{aligned}$$

and the second integral equals

$$\lambda(V)^{-1} \int_G f(y) \int_{V\times G} |h \otimes k|^2(ty, z)\, d(t, z)\, dy.$$

Similarly,

$$|M(L_x f) - \lambda(W)^{-1} \int_G f(y) \int_{W\times G} |L_{x^{-1}} h \otimes k|^2(ty, z)\, d(t, z)\, dy| < \delta.$$

Since $|\lambda(V)^{-1} - \lambda(W)^{-1}| < \varepsilon\lambda(V)^{-1}$, we get (putting

$$a(y)^2 = \int_{W\times G} |L_{x^{-1}} h \otimes k|^2(ty, z)\, d(t, z)$$

and

$$b(y)^2 = \int_{V\times G} |h \otimes k|^2(ty, z)\, d(t, z))$$

$$|M(L_x f - f)| \leq 2\delta + \|f\|_\infty(\varepsilon + \lambda(W)^{-1}) \int_G |a(y)^2 - b(y)^2|\, dy.$$

Now we apply Lemma 4. We have $\int_G a(y)^2\, dy = \lambda(W)$, $\int_G b(y)^2\, dy = \lambda(V)$. Furthermore,

$$a(y) = \Delta(y)^{-1/2} \|L_{x^{-1}} h \otimes k | Wy \times G\|_2 \quad \text{and} \quad b(y) = \Delta(y)^{-1/2} \|h \otimes k | Vy \times G\|_2.$$

(Δ denotes the Haar modulus of G.)

By our assumptions on T, we have

$$\|h \otimes k|Vy \times G\|_2 \geq \|T(h \otimes k)|Vy \times G\|_2 \geq \|T(h \otimes k)|Wy \times G\|_2.$$

(It follows easily from [**E**, 4.8, p. 226] that $\operatorname{supp}(T(h \otimes k)) \subseteq \operatorname{supp} T \cdot \operatorname{supp}(h \otimes k)$, where $\cdot$ refers to the diagonal action of G on $G \times G$.) This gives

$$\begin{aligned}\int_G a(y)b(y)\,dy &\geq \int_G \|L_{x^{-1}}h \otimes k|Wy \times G\|_2 \|T(h \otimes k)|Wy \times G\|_2 \Delta(y^{-1})\,dy \\ &\geq \int_G \int_{Wy \times G} |(L_{x^{-1}}h \otimes k) \cdot T(h \otimes k)|(t,z)\,d(t,z)\,\Delta(y^{-1})\,dy \\ &= \int_G \int_{W \times G} |(L_{x^{-1}}h \otimes k) \cdot T(h \otimes k)|(ty,z)\,d(t,z)\,dy \\ &\geq \lambda(W)|(T(h \otimes k), L_{x^{-1}}h \otimes k)|.\end{aligned}$$

Hence, by Lemma 4,

$$\begin{aligned}\int_G |a(y)^2 - b(y)^2|\,dy &\leq ((\lambda(W) + \lambda(V))^2 - 4\lambda(W)^2 |(T(h \otimes k), L_{x^{-1}}h \otimes k)|^2)^{1/2} \\ &\leq 2\lambda(W)(1 + 2\varepsilon - |(T(h \otimes k), L_{x^{-1}}(h \otimes k)|^2)^{1/2} \\ &\leq \lambda(W)(2 + 2\varepsilon - |(T(h \otimes k), L_{x^{-1}}h \otimes k)|^2)\end{aligned}$$

and the result follows.

If M is a mean on $L^\infty(G)$ or $C_{\mathrm{lu}}(G)$, $x \in G$, we write

$$d(M,x) = \sup\{|M(L_x f - f)| : \|f\|_\infty \leq 1\}.$$

PROPOSITION 2. *Assume there exists $c < 2$ such that for each $x_1, \ldots, x_n \in G$ (not necessarily distinct points) there exists a mean M on $C_{\mathrm{lu}}(G)$ with $\sum_{i=1}^n d(M,x_i) \leq cn$. Then G is amenable.*

PROOF. This will be done in two steps. Let $\mathcal{M}$ be the set of means on $C_{\mathrm{lu}}(G)$. First we show that, given $x_1, \ldots, x_n \in G$, there exists $M \in \mathcal{M}$ such that $d(M,x_i) \leq c$ for $i = 1, \ldots, n$.

To each $M \in \mathcal{M}$ we associate the n-tuple $(d(M,x_i))_{i=1}^n \in \mathbf{R}^n$. Let C be the convex hull of these elements, take $\varepsilon > 0$ and assume that C contains no vector (t_i) with $\|(t_i)\|_\infty \leq c + \varepsilon$. Then we apply the separation theorem for convex sets. There exists $(u_i) \in \mathbf{R}^n$ such that $\|(u_i)\|_1 = 1$ and $\sum_{i=1}^n t_i u_i \geq c + \varepsilon$ for all $(t_i) \in C$. We may assume that $u_i \geq 0$ and $u_i = p_i/q \in \mathbf{Q}$ for $i = 1, \ldots, n$ (replacing ε by $\varepsilon/2$). Then $\sum p_i = q$. Put $y_1 = \cdots = y_{p_1} = x_1$, $y_{p_1+1} = \cdots = y_{p_1+p_2} = x_2$, etc. Then, by assumption, there exists $M \in \mathcal{M}$ such that $\sum_{j=1}^q d(M,y_j) \leq cq$. But

$$q^{-1} \sum_{j=1}^q d(M,y_j) = q^{-1} \sum_{i=1}^n p_i d(M,x_i) = \sum_{i=1}^n u_i d(M,x_i) \geq c + \frac{\varepsilon}{2}$$

by definition of (u_i), and we arrive at a contradiction. Thus, there exists $(t_i) \in C$ with $\|(t_i)\|_\infty \leq c + \varepsilon$. Then $t_i = \sum \lambda_k d(M_k, x_i)$ for some $M_k \in \mathcal{M}$, $\lambda_k \geq 0$ with $\sum \lambda_k = 1$. Put $M = \sum \lambda_k M_k$. Then $M \in \mathcal{M}$ and $d(M,x_i) \leq t_i \leq c + \varepsilon$ for all i. Since $\varepsilon > 0$ was arbitrary, the first step follows from the compactness of $\mathcal{M}$.

The second step is now to show that, given $x_1, \dots, x_n \in G$, there exists a mean M on $L^\infty(G)$ such that $d(M, x_i) \le c$ for $i = 1, \dots, n$. Then the result will follow from [**Lo1**, Lemma 3 and Theorem 1 (see also [**Gi**, **Lep 1**]). The extension from $C_{\mathrm{lu}}(G)$ to $L^\infty(G)$ is done essentially as in [**Gr**, p. 28]. Fix $u \in L^1(G)$ with $u \ge 0$, $\|u\|_1 = 1$. For $M \in \mathcal{M}$, $f \in L^\infty(G)$, put $M_1(f) = M(u * f)$. Then M_1 is a mean on $L^\infty(G)$.

Given $x_1, \dots, x_n \in G$, we have $M_1(L_{x_i} f - f) = M((u * \delta_{x_i} - u) * f)$. For $\varepsilon > 0$ there exists $v \in L^1(G)$ with $v \ge 0$, $\|v\|_1 = 1$, such that $\|(u * \delta_{x_i} - u) * (v - \delta_e)\|_1 < \varepsilon$ for $i = 1, \dots, n$. Then we can find $y_j \in G$, $\lambda_j \ge 0$ with $\sum \lambda_j = 1$ such that $\|(u - \sum \lambda_j \delta_{y_j}) * (\delta_{x_i} * v - v)\|_1 < \varepsilon$. This gives combined (assuming $\|f\|_\infty \le 1$) :

$$\begin{aligned} |M_1(L_{x_i} f - f)| &< 2\varepsilon + \sum \lambda_j |M(\delta_{y_j} * (\delta_{x_i} * v - v) * f)| \\ &= 2\varepsilon + \sum \lambda_j |M((\delta_{y_j x_i y_j^{-1}} - \delta_e) * \delta_{y_j} * v * f)| \\ &\le 2\varepsilon + \sum \lambda_j d(M, y_j x_i y_j^{-1}). \end{aligned}$$

Choosing $M \in \mathcal{M}$ so that $d(M, y_j x_i y_j^{-1}) \le c$ for all i, j, we get $d(M_1, x_i) \le c + 2\varepsilon$ for all i.

PROOF OF THEOREM 1. If G is amenable, then (a)⇒(b) by [**De**, Theorem 9; **H 2**, Theorem 1; or **R**, Theorem 1]. (b)⇒(c) is trivial. Now assume that $\sup\{\|uv\|_A \colon \|v\|_A \le 1\} > c\|u\|_A$ for each $u \in A(G)$ (where $c > 0$). The discrete case can be settled as in [**N**]: Take $h \in l^2(G)$. By Lemma 2 of [**N**] there is $\Phi \in l^\infty(G)$ such that $\|\Phi\|_\infty = 1$ and $\|h\Phi\|_A \ge c_1 \|h\|_2$. By assumption there is $u \in A(G)$ such that $\|u\|_A = 1$ and $\|uh\Phi\|_A \ge c\|h\Phi\|_A \ge cc_1\|h\|_2$. But $\|uh\Phi\|_A \le \|uh\Phi\|_2 \le \|uh\|_2$. c, c_2 are independent of u, Φ and therefore $l^2(G)$ is closed in $M(A(G), l^2(G))$. Now the same proof as for Theorem 1 of [**N**] shows that G is amenable.

Now assume G is nondiscrete. Take $x_1, \dots, x_n \in G$ and put $\varepsilon = c^2/200$. If V is a neighbourhood of e, choose neighbourhoods U, W such that $\lambda(W) > (1+\varepsilon)^{-1}\lambda(V)$ and $UW \subseteq V$. Take $u \in A(G) \cap P_1(G)$ with $\operatorname{supp} u \in U$. By Lemma 3 there exists $v \in A(G) \cap P_1(G)$ such that

$$\sum_{i=1}^n \|v \cdot L_{x_i} u\|_A > \frac{cn}{4}.$$

Put $c_i = \|v \cdot L_{x_i} u\|_A$, $v(x) = (L_x h, h)$, $u(x) = (L_x k, k)$, where $h,\ k \in L^2(G)$, $\|h\|_2 = \|k\|_2 = 1$. Then there exists $T_i' \in \mathrm{VN}(G)$ such that $\|T_i'\| \le 1$ and $\langle T_i', v \cdot L_{x_i} u\rangle = c_i$. We have $\langle L_{x_i^{-1}} T_i', L_{x_i^{-1}} v \cdot u\rangle = \langle T_i', v \cdot L_{x_i} u\rangle$. Put $w(x) = \lambda(W)^{-1}\lambda(xW \cap V)$. Then $w = 1$ on U and $\|w\|_A \le (\lambda(V)/\lambda(W))^{1/2} \le 1 + \varepsilon$. Put $T_i = w \cdot (L_{x_i^{-1}} T_i') \cdot (1+\varepsilon)^{-1}$. Then $\|T_i\| \le 1$,

$$(T_i(h \otimes k),\ L_{x_i^{-1}} h \otimes k) = \langle T_i, L_{x_i^{-1}} v \cdot u\rangle = c_i(1+\varepsilon)^{-1} \ge c_i/2$$

(since $\operatorname{supp} u \subseteq U$). We define a mean M on $C_{\mathrm{lu}}(G)$ by $M(f) = \int_G f(x)|h(x)|^2\,dx$. By Lemma 5,

$$|M(L_{x_i} f - f)| \le 2\sup\{\|L_y f - f\| \colon y \in V\} + \|f\|_\infty (2 + 3\varepsilon - c_i^2/4).$$

We have $\sum c_i \ge nc/4$, hence $\sum c_i^2 \ge n(c/4)^2$. Thus

$$\sum_{i=1}^n |M(L_{x_i} f - f)| \le 2\sup\{\|L_y f - f\| \colon\ y \in V\} + n\|f\|_\infty \left(2 + 3\varepsilon - \frac{c^2}{64}\right).$$

The neighbourhood V was arbitrary. Considering a limit along the neighbourhood filter, we get a mean M on $C_{\mathrm{lu}}(G)$ such that $\sum_{i=1}^n d(M, x_i) \le n(2 + 3\varepsilon - c^2/64)$. Now Proposition 2 shows that G is amenable.

A related criterion for amenability can be obtained using the space $S_0(G)$ (see [**Fe**]). It can be defined as follows: Let (ψ_i) be a bounded uniform partition of unity in $A(G)$, i.e. all ψ_i are uniformly bounded in $A(G)$, $\operatorname{supp}\psi_i$ is contained in a left-translate of a fixed compact subset K and there exists n_0 such that for each $x \in G$ the supports of at most n_0 functions ψ_i intersect xK (e.g. for $G = \mathbf{R}$ one can take trapezoidal functional of width 2 with integer endpoints; see also [**Lo 2**, Proposition 1, p. 131]). Then $S_0(G) = \{u : G \to \mathbf{C} : \|u\|_{S_0} = \sum \|u\psi_i\|_A < \infty\}$.

THEOREM 2. *G is amenable iff* $M(A(G),\ S_0(G)) = S_0(G)$.

PROOF. Let G be amenable and consider $\Phi \in M(A(G), S_0(G))$. Given finitely many ψ_i $(i \in I_0)$, $\varepsilon > 0$, there exists $u \in A(G)$ such that $\|u\|_A < 1+\varepsilon$, $u(x) = 1$ for $x \in \operatorname{supp}\psi_i$ $(i \in I_0)$. Hence $u\Phi \in S_0(G)$ and $\sum_{i\in I_0} \|\Phi\psi_i\|_A = \sum_{i\in I_0} \|u\Phi\psi_i\|_A \le \|\Phi\|_M(1+\varepsilon)$ (where $\|\ \|_M$ denotes the multiplier norm). Hence $\Phi \in S_0(G)$.

Conversely, assume that $\sup\{\|u\Phi\|_{S_0}\colon \|u\|_A \le 1\} > c\|\Phi\|_{S_0}$ for all $\Phi \in S_0(G)$. Then we proceed as in Theorem 1. Choose $u \in A(G) \cap P_1(G)$ and take $x_1, \ldots, x_n \in G$. As in Lemma 3, there is $v \in S_0(G)$, such that $\sum_j \|v\cdot L_{x_j} u\|_{S_0} > cn/2$, $\|v\|_{S_0} \le 1$. Since $S_0(G)$ is contained in $A(G)$, we have $\|v\|_A \le c_1$. For functions whose support is contained in a translate of a fixed compact set, the A- and S_0-norms are equivalent [**Fe**, p. 275]. Thus we get $\sum_j \|v \cdot L_{x_j} u\|_A > cc_2 n/2$. Now continue as in Lemma 3 and Theorem 1. In the discrete case, $S_0(G) = l^1(G)$, hence one can use [**N**, Theorem 1].

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