NONCONTRACTIVE UNIFORMLY LIPSCHITZIAN SEMIGROUPS IN HILBERT SPACE

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ABSTRACT. It is shown that any \( k \)-Lipschitzian, \( k < \pi/2 \), noncontractive commutative semigroup acting on a closed bounded convex set in Hilbert space has a common fixed point.

1. Introduction. Let \( \mathcal{F} = \{ f_{\alpha} | \alpha \in A \} \) be a semigroup of mappings of a metric space \( (M, d) \) into itself. The semigroup \( \mathcal{F} \) is said to have a fixed point if there exists \( x_0 \in M \) with \( f_{\alpha}(x_0) = x_0 \) for all \( \alpha \in A \), and \( \mathcal{F} \) is said to be uniformly \( k \)-Lipschitzian if for each \( x, y \in M \) and each \( \alpha \in A \),

\[
d(f_{\alpha}(x), f_{\alpha}(y)) \leq kd(x, y).
\]

\( \mathcal{F} \) is said to be left reversible if every two right ideals of \( \mathcal{F} \) have a nonempty intersection (i.e. for \( f, g \in \mathcal{F}, f \mathcal{F} \cap g \mathcal{F} \neq \emptyset \)). Commutative semigroups, and in particular \( \{ f^n : n = 0, 1, \ldots \} \) for some function \( f \), are left reversible.

In [4] Goebel, Kirk, and Thele showed that if \( X \) is a Banach space with \( \delta(1) > 0 \) (where \( \delta \) is the modulus of convexity function) then there is a constant \( k'_0 > 1 \) such that any left reversible uniformly \( k \)-Lipschitzian semigroup \( \mathcal{F}, k < k'_0 \), acting on a closed bounded convex set \( K \) in \( X \) has a fixed point. Clearly there is a maximum choice of \( k'_0 \) which we call \( k_0 \). In [4] it was shown that for Hilbert space \( (\mathcal{H}), \sqrt{\frac{5}{2}} \leq k_0 \leq 2 \). Downing and Ray [3] improved this estimate of \( k_0 \), for Hilbert space, by showing that \( \sqrt{2} \leq k_0 \) while in [1], Baillon has an example (presented here in §2) which shows that \( k_0 \leq \pi/2 \). (Lim [6 and 7] has improved the results of [4] for the \( L^p \) spaces.)

In this note we show that under the additional assumption that \( \mathcal{F} \) be a noncontractive (i.e. \( \|x - y\| \leq \|f(x) - f(y)\| \) for each \( f \in \mathcal{F} \) and all \( x, y \in K \)) uniformly \((\pi/2 - \delta)\)-Lipschitzian (\( \delta > 0 \)) commutative semigroup, then \( \mathcal{F} \) has a fixed point. The example of Baillon, showing that \( k_0 \leq \pi/2 \), is noncontractive (this is proven in §2 of this paper), hence \( \pi/2 \) is the exact value of \( k_0 \) in the case of noncontractive commutative semigroups. We note that the example in [4] showing that \( k_0 \leq 2 \) is also noncontractive.

2. In this section we present Baillon’s example [1] of a noncontractive uniformly \( \pi/2 \)-Lipschitzian semigroup of mappings of a closed bounded convex subset of \( l^2 \) which contains no fixed point.
Let $S$ be the shift operator. That is, $S((x_1, x_2, \ldots)) = (0, x_1, x_2, \ldots)$. Let $e_1 = (1, 0, 0, \ldots)$ and $K = \{x \in l^2: x = (x_1, x_2, \ldots), \|x\| \leq 1, x_i \geq 0, i = 1, 2, \ldots\}$. Define $f$ by

$$f(x) = \cos \left(\frac{\pi}{2} \|x\|\right) e_1 + \sin \left(\frac{\pi}{2} \|x\|\right) \frac{S(x)}{\|x\|}.$$ 

Several properties of $f$ are immediate,

(2.1) $\|f(x)\| = 1$ for all $x \in K$.

(2.2) If $\|x\| = 1$ then $f(x) = S(x)$.

(2.3) $f^n(x) = S^{n-1}f(x)$ (from 2.1 and 2.2).

If $f(x) = x(x \in K)$ then $x = f^2(x) = Sf(x) = S(x)$. Since $S(x) = x$ implies that $x = 0$, and $f(0) = (1, 0, \ldots)$, $f$ has no fixed points. Once it is established that

(2.4) $\|x - y\| \leq \|f(x) - f(y)\| < \pi/2\|x - y\|, \quad x, y \in K$,

the semigroup $\mathcal{F} = \{f^n \mid n = 1, 2, \ldots\}$ is readily seen to be noncontractive uniformly $\pi/2$-Lipschitzian with no fixed points. Note that the second inequality of 2.4 is strict, so $\pi/2$ is a strict Lipschitz constant for $\mathcal{F}$.

The fact that $\mathcal{F}$ is noncontractive has not been mentioned in the literature, we believe, and as it shows that the constant $\pi/2$ in our main theorem is the “best possible”, we present a brief proof (due to the referee).

**Proof that $\mathcal{F}$ is noncontractive.** We readily compute that

$$\|f(x) - f(y)\|^2 - \|x - y\|^2 = \left(\cos \left(\frac{\pi}{2} \|x\|\right) - \cos \left(\frac{\pi}{2} \|y\|\right)\right)^2$$

$$+ \left\| \frac{x}{\|x\|} \sin \left(\frac{\pi}{2} \|x\|\right) - \frac{y}{\|y\|} \sin \left(\frac{\pi}{2} \|y\|\right) \right\|^2 - \|x - y\|^2$$

$$= 2 - 2 \cos \left(\frac{\pi}{2} \|x\|\right) \cos \left(\frac{\pi}{2} \|y\|\right) - (\|x\|^2 + \|y\|^2)$$

$$+ 2 \left(\frac{x}{\|x\|}, \frac{y}{\|y\|}\right) \left\{ \|x\| \|y\| - \sin \left(\frac{\pi}{2} \|x\|\right) \sin \left(\frac{\pi}{2} \|y\|\right) \right\}.$$

The well-known inequality $\sin t \geq 2t/\pi$ (for $0 \leq t \leq \pi/2$) shows that the expression in curly brackets is less than or equal to 0, thus replacing $\langle x/\|x\|, y/\|y\|\rangle$ by one, we get

$$\|f(x) - f(y)\|^2 - \|x - y\|^2 \geq 2 - 2 \cos \frac{\pi}{2} (\|x\| - \|y\|) - (\|x\| - \|y\|)^2$$

$$= 4 \sin^2 \frac{\pi}{4} (\|x\| - \|y\|) - (\|x\| - \|y\|)^2.$$

A final appeal to $\sin t \geq 2t/\pi$ shows that this is nonnegative. \( \square \)

**3.** In this section $\mathcal{H}$ shall denote a Hilbert space, $K$ a closed bounded convex set of $\mathcal{H}$, $S$ a subset of $K$, $f: K \to K$ a noncontractive function, and $\mathcal{F}: K \to K$ a noncontractive commutative semigroup. We show that if $\mathcal{F}$ is uniformly $(\pi/2 - \delta)$-Lipschitzian, $\delta > 0$, then $\mathcal{F}$ has a fixed point.

Let $S \subseteq K$. Then $S$ is bounded, as $K$ is. For $x \in \mathcal{H}$ define

(3.1) $r(S, x) = \sup \{\|x - s\|: s \in S\},$
and let \( c(S) \) be the unique point of \( \mathcal{H} \) such that \( r(S, c(S)) = r(S) \). Equivalently, \( c(S) \) is the unique point of \( \mathcal{H} \) such that \( \overline{B}(c(S), r(S)) \supseteq S \) (where \( \overline{B}(x, r) \) denotes the closed ball about \( x \) of radius \( r \)). The point \( c(S) \) is called the Chebyshev center of \( S \). When no confusion arises, let \( r(x) = r(S, x) \), \( r = r(S) \) and \( c = c(S) \). It is well known and easily shown (cf. [5]) that \( c(S) \) lies in the closed convex hull of \( S \) (denoted \( \overline{\text{co}}(S) \)) and hence is in \( K \).

**Lemma 3.1.** For \( x \in \mathcal{H} \), \( r^2 + \|c - x\|^2 \leq r^2(x) \).

**Proof.** To simplify notation, assume that \( c = 0 \). Then \( \|y\| \leq r \) for all \( y \in S \). For a contradiction, suppose that

\[
\varepsilon = \frac{r^2 + \|x\|^2 - r^2(x)}{2\|x\|} > 0
\]

and let \( z = \varepsilon x/\|x\| \). We claim, then, that \( r^2(z) \leq r^2 - \varepsilon^2 \). To see this let \( y \in S \), and consider the two cases \( \langle y, x/\|x\| \rangle > \varepsilon \) and \( \langle y, x/\|x\| \rangle \leq \varepsilon \). In the first case a straightforward calculation shows that \( \|y - z\|^2 \leq r^2 - \varepsilon^2 \). For the second case the cosine law shows that

\[
\|y - z\|^2 = \|y - x\|^2 - \|x - z\|^2 + 2\langle y - z, x - z \rangle.
\]

It can now be shown that \( \langle y - z, x - z \rangle \leq 0 \), hence

\[
\|y - z\|^2 \leq r^2(x) - \|x - z\|^2.
\]

Using the definition of \( \varepsilon \) this then yields

\[
\|y - z\|^2 \leq r^2 - \varepsilon^2.
\]

Thus \( r^2(z) \leq r^2 - \varepsilon^2 \). However this contradicts the definition of \( c \), and hence the lemma has been established.

**Lemma 3.2.** For every \( S \subseteq K \), \( \sup\{r(T): T \subseteq S \text{ and } T \text{ is finite} \} = r \).

**Proof.** Assume there is an \( r' < r \) such that \( r(T) < r' \) for every finite set \( T \subseteq S \). The finite intersection property then shows that \( \bigcap_{x \in S} \overline{B}(x, r') \neq \emptyset \), implying that \( r = r(s) \leq r' \) a contradiction.

**Theorem 3.3 (Kirszbraun).** Let \( \{x_i: i \in I\} \) and \( \{y_i: i \in I\} \) be sets in \( \mathcal{H} \) and \( \{r_i: i \in I, r_i > 0\} \) be a set of real numbers such that

\[
\|x_i - x_j\| \leq \|y_i - y_j\| \quad (i, j \in I).
\]

Then

\[
\bigcap_{i \in I} \overline{B}(x_i, r_i) = \emptyset \quad \text{implies} \quad \bigcap_{i \in I} \overline{B}(y_i, r_i) = \emptyset.
\]

**Proof.** [8, p. 47].

**Lemma 3.4.** For each \( S \subseteq K \), \( r(S) \leq r(f(S)) \).

**Proof.** For a contradiction, assume that \( \varepsilon = r(S) - r(f(S)) > 0 \). By the definition of \( r(S) \),

\[
\bigcap_{s \in S} \overline{B}(s, r(S) - \varepsilon) = \emptyset.
\]
As $f$ is noncontractive,
\[ \|f(s_1) - f(s_2)\| \geq \|s_1 - s_2\| \quad \text{for all } s_1, s_2 \in S. \]
Kirszbraun's theorem implies that
\[ \emptyset = \bigcap_{s \in S} \overline{B}(f(s), r(S) - \epsilon) = \bigcap_{s \in S} \overline{B}(f(S), r(f(S))) = \{c(f(S))\} \]
leading to a contradiction. \(\square\)

**Lemma 3.5.** Assume that $S \subseteq K$ satisfies $f(S) \subseteq S$. Then $r(S) = r(f(S))$ and $c(S) = c(f(S))$.

**Proof.** Because $f(S) \subseteq S$ we have $r(f(S)) \leq r(S)$. On the other hand Lemma 3.4 shows that $r(S) \leq r(f(S))$. Hence $r(S) = r(f(S))$. Now
\[ f(S) \subseteq S \subseteq \overline{B}(c(S), r(S)) = \overline{B}(c(S), r(f(S))). \]
However $c(f(S))$ is the unique point in $\mathcal{H}$ such that $\overline{B}(c(f(S)), r(f(S))) \subseteq f(S)$. Thus $c(S) = c(f(S))$. \(\square\)

**Lemma 3.6.** Let $S \subseteq K$ satisfy $f(S) \subseteq S$. Then for each $x \in K$, $\|f(x) - c(S)\| \geq \|x - c(S)\|$.

**Proof.** From Lemma 3.5, $c(S) = c(f(S))$ and $r(S) = r(f(S))$. Letting $c = c(S)$ and $r = r(S)$, assume for some $x \in K$ that $\|x - c\| > \|f(x) - c\|$. Let $\epsilon = \|x - c\| - \|f(x) - c\|$. Now $\bigcap_{s \in S} \overline{B}(s, r) = \{c\}$, so
\[ \phi = \bigcap_{s \in S} \overline{B}(s, r) \cap \overline{B}(x, \|x - c\| - \epsilon). \]
By Kirszbraun's Theorem (Theorem 3.3)
\[ \phi = \bigcap_{s \in S} \overline{B}(f(s), r) \cap \overline{B}(f(x), \|f(x) - c\|) = \{c\}. \]

Thus we have a contradiction, hence
\[ \|x - c\| \leq \|f(x) - c\|. \quad \square \]

If $\mathcal{F} = \{f_\alpha : \alpha \in A\}$ is a commutative (or left reversible) semigroup then the set $A$ can be directed as follows. For $\alpha, \beta \in A$ define
\[ \alpha \leq \beta \quad \text{if and only if } f_\alpha \mathcal{F} \supseteq f_\beta \mathcal{F}. \]
Thus for each $x$, $\mathcal{F}(x) = \{f_\alpha(x) : \alpha \in A\}$ is a net. Also, without loss of generality when finding fixed points, it may be assumed the identity is in $\mathcal{F}$.

**Corollary 3.7.** If $\mathcal{F} : K \to K$ is a commutative semigroup of noncontractive mappings, $K$ a closed bounded convex set in $\mathcal{H}$, and $\mathcal{F} : S \subseteq K \to S$ then for $\alpha \geq \beta$,
\[ \|f_\alpha(x) - c(S)\| \geq \|f_\beta(x) - c(S)\| \]
for each $x \in K$.

**Proof.** Because $f_\alpha \mathcal{F} \supseteq f_\beta \mathcal{F}$, and the identity is in $\mathcal{F}$, $f_\beta = f_\alpha f_\eta$, $\eta \in A$, and as $\mathcal{F}$ is commutative, $f_\beta(x) = f_\eta(f_\alpha(x))$. The corollary follows from Lemma 3.6. \(\square\)

**Remark.** This corollary is the only point in our argument where the commutativity of $\mathcal{F}$ is used. If $\mathcal{F}$ were right-reversible, rather than commutative, the corollary would be valid.
**Lemma 3.8.** For every \( \varepsilon > 0 \) there is a finite subset \( T \subseteq S \) such that \( \|c(S) - c(f(T))\| < \varepsilon \) for every noncontractive function \( f \) with \( f(S) \subseteq S \).

**Proof.** Let \( \varepsilon > 0 \) be given. Lemma 3.2 shows that there is a finite subset \( T \subseteq S \) with \( r^2(S) - r^2(T) < \varepsilon^2 \). Let \( f \) be an arbitrary noncontractive function with \( f(S) \subseteq S \). From Lemma 3.1, we have

\[
r^2(f(T)) + \|c(f(T)) - c(S)\|^2 \leq r^2(f(T), c(S)) \leq r^2(S).
\]

Lemma 3.4 shows that \( r(T) \leq r(f(T)) \), so

\[
r^2(T) + \|c(S) - c(f(T))\|^2 \leq r^2(S).
\]

Hence \( \|c(S) - c(f(T))\| < \varepsilon \), establishing the lemma. \( \square \)

**Lemma 3.9.** Let \( \varepsilon > 0 \) be given. Then there is a finite subset \( T \subseteq S \) such that for every linear functional \( h, \|h\| = 1 \), and each noncontractive function \( f \) satisfying \( f(S) \subseteq S \) it is true that for some \( x \in T \),

\[
h(f(x)) - h(c(S)) < \varepsilon.
\]

**Proof.** By Lemma 3.8 there is a finite subset \( T \subseteq S \) such that for every noncontractive function \( f \), \( \|c(S) - c(f(T))\| < \varepsilon \). As has been mentioned, \( c(f(T)) \in \overline{co}(f(T)) \), hence for some \( x \in T \), \( h(x) - h(c(f(T))) \leq 0 \). Thus

\[
h(x) - h(c(S)) \leq h(c(f(T))) - h(c(S)) \leq \varepsilon.
\]

\( \square \)

By an arc \( \gamma \) in \( \mathcal{H} \) (or any metric space) we mean the image of a function \( \gamma: [a, b] \rightarrow \mathcal{H} \) for some interval \( [a, b] \) of \( \mathbb{R} \). The length of an arc may be defined by purely metric means, without any differentiability assumption. We refer the reader to a book on metric geometry such as [2] for a discussion of arcs and their lengths. Let \( l(\gamma) \) denote the arclength of (the image of) \( \gamma \). If \( f \) satisfies

\[
k_1\|x - y\| \leq \|f(x) - f(y)\| \leq k_2\|x - y\|
\]

and \( \gamma \) lies in the domain of \( f \), then \( f \circ \gamma \) is an arc and

\[
k_1l(\gamma) \leq l(f(\gamma)) \leq k_2l(\gamma).
\]

**Lemma 3.10.** Let \( \gamma(t), a \leq t \leq b \) be an arc satisfying

\[
\frac{\langle \gamma(a), \gamma(b) \rangle}{\|\gamma(a)\| \|\gamma(b)\|} \leq \cos \theta, \quad 0 \leq \theta \leq \pi,
\]

and \( \|\gamma(t)\| \geq d, t \in [a, b] \). Then the length of \( \gamma \) is at least \( d \cdot \theta \).

**Proof.** Let \( P \) be the projection of \( \mathcal{H} \setminus \{0\} \) onto the sphere of radius \( d \), defined by \( P(x) = d \cdot x/\|x\| \). Certainly \( P \) is nonexpansive on \( \{x \in \mathcal{H}: \|x\| \geq d\} \) so \( l(P(\gamma)) \leq l(\gamma) \). However \( P(\gamma) \) is no shorter than the geodesic on \( S \) joining \( P(\gamma(a)) \) to \( P(\gamma(b)) \), whose length is at least \( d \cdot \theta \).

\( \square \)

**Theorem.** Let \( \mathcal{F} = \{f_\alpha: \alpha \in A\} \) be a commutative semigroup (with identity) of self-mappings of a closed bounded convex set \( K \) in a Hilbert space \( \mathcal{H} \). Assume that for some \( \delta > 0 \),

\[
\|x - y\| \leq \|f_\alpha(x) - f_\alpha(y)\| \leq (\pi/2 - \delta)\|x - y\|.
\]

Then \( \mathcal{F} \) has a fixed point.
PROOF. Let \( x_0 \in K \) be arbitrary, \( x_\alpha = f_\alpha(x_0) \) and \( S = \{x_\alpha | \alpha \in A\} \). Note that \( f_\alpha(S) \subseteq S \) for all \( \alpha \in A \). Let \( c = c(S) \), \( r = r(S) \). As mentioned earlier, \( c \in \overline{c_0\{x_\alpha: \alpha \in A\}} \subseteq K \) (see [5]). Let \([c, x_\alpha]\) denote the arc \( \{y: y = \lambda(c) + (1-\lambda)x_\alpha, 0 \leq \lambda \leq 1\} \). We now show that for some \( x \in \bigcup_{\alpha \in A}[c, x_\alpha], \|c - f_\alpha(x)\| \leq (1 - \delta/\pi)r \), for all \( \alpha \in A \).

Clearly if \( r = 0 \), then \( x \) may be taken to be \( c \) (which is a fixed point of \( F \)). Thus, for a contradiction, assume that \( r > 0 \) and that \( \sup\{\|c - f_\alpha(x)\|: \alpha \in A\} > (1 - \delta/\pi)r \) for all \( x \in \bigcup_{\alpha \in A}[c, x_\alpha] \).

Let \( \epsilon > 0 \) be given. By Lemma 3.9 choose a finite set \( T \subseteq S \) such that for any \( \alpha \in A \), and any linear functional \( h, \|h\| = 1 \), there is an \( z \in T \) with \( h(f_\alpha(z)) - h(c) < \epsilon \). Since \( \bigcup_{y \in T}[c, y] \) is compact, Corollary 3.7 implies that for some \( \eta \in A \) and for every \( x \in \bigcup_{y \in T}[c, y] \),

\[
\|c - f_\eta(x)\| > \left(1 - \frac{\delta}{\pi}\right)r.
\]

Define \( h(x) = \frac{f_\eta(x) - c}{\|f_\eta(x) - c\|} \cdot x \) and let \( z \in T \) satisfy \( h(f_\eta(z)) - h(c) < \epsilon \). Thus

\[
\left(\frac{f_\eta(c) - c}{\|f_\eta(c) - c\|}, \frac{f_\eta(z) - c}{\|f_\eta(z) - c\|}\right) \cdot \frac{\epsilon}{(1 - \delta/\pi)r}.
\]

Hence angle \( f_\eta(c)c_f_\eta(z) \) is greater than \( \cos^{-1}(\epsilon/(1 - \delta/\pi)r) \) and by Lemma 3.10

\[
l(f_\eta[c, z]) > \left(1 - \frac{\delta}{\pi}\right)r \cos^{-1}\left(\frac{\epsilon}{(1 - \delta/\pi)r}\right).
\]

As \( \epsilon > 0 \) was arbitrary, it must be that

\[
(3.3) \quad \sup\{l(f_\eta[c, z]): \eta \in A, z \in S\} \geq \left(\frac{\pi}{2} - \frac{\delta}{2}\right)r.
\]

However, as \( \|f_\eta(x) - f_\eta(y)\| \leq \left(\frac{\pi}{2} - \delta\right)\|x - y\| \) for all \( x, y \in K \) and \( \eta \in A \), then for each \( \eta \in A \) and \( z \in S \),

\[
(3.4) \quad l(f_\eta[c, z]) \leq \left(\frac{\pi}{2} - \delta\right)\|c - z\| \leq \left(\frac{\pi}{2} - \delta\right)r.
\]

Combining 3.3 and 3.4, we reach the desired contradiction, and conclude that for some \( x \in \bigcup_{\alpha \in A}[c, x_\alpha] \),

\[
\|c - f_\alpha(x)\| \leq \left(1 - \frac{\delta}{\pi}\right)r \quad \text{for all } \alpha \in A.
\]

Let \( y_0 \in K \) be arbitrary and let \( r = r(\{f_\alpha(y_0): \alpha \in A\}) \). Assume that for \( n < k \), \( y_n \) has been defined so that

\[
(3.5) \quad \|y_n - y_{n-1}\| \leq 2 \left(1 - \frac{\delta}{\pi}\right)^{n-1} r
\]

and

\[
(3.6) \quad r(\{f_\alpha(y_n): \alpha \in A\}) \leq \left(1 - \frac{\delta}{\pi}\right)^n r.
\]
Using the above argument with $x_0 = y_{k-1}$ we find a point $y_k = x$ such that

\begin{equation}
\{f_\alpha(y_k): \alpha \in A\} \subseteq B \left( c \left( \{f_\alpha(y_{k-1}): \alpha \in A\} \right), \left( 1 - \frac{\delta}{\pi} \right)^k r \right),
\end{equation}

and hence (3.6) is satisfied with $n = k$, and

\begin{equation}
\|y_k - c\{f_\alpha(y_{k-1}): \alpha \in A\}\| \leq \left( 1 - \frac{\delta}{\pi} \right)^k r \leq \left( 1 - \frac{\delta}{\pi} \right)^{k-1} r
\end{equation}

(recall we assume that $\mathcal{F}$ contains the identity). Also,

\begin{equation}
\|y_{k-1} - c\{f_\alpha(y_{k-1}): \alpha \in A\}\| \leq r(\{f_\alpha(y_{k-1}): \alpha \in A\}) \leq \left( 1 - \frac{\delta}{\pi} \right)^{n-1} r.
\end{equation}

Hence 3.8 and 3.9 together show 3.5 is satisfied with $n = k$.

We conclude, using the continuity of the $f_\alpha$, that $\{y_n\}$ is a Cauchy sequence, and its limit is a fixed point of $\mathcal{F}$. □

**Remark.** The above theorem holds for a right reversible, rather than commutative, semigroup. See the remark following Corollary 3.6.

**References**

7. ———, *Some inequalities and their applications to fixed point theorems of uniformly Lipshitzian mappings in $L^p$ spaces*, preprint.