FUNCTIONALS OF RATIONAL TYPE
OVER THE CLASS S
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ABSTRACT. Let $L$ be a continuous linear functional on the space of functions holomorphic in the unit disk, and let $f$ be a function in the class $S$ for which $\Re L$ achieves its maximum on $S$. Then $L$ is said to be of rational type if the expression $L(f^2/(f - w))$, which occurs in Schiffer's differential equation, is a rational function of $w$. Various equivalent formulations of "rational type" are found and an application to the process of arc truncation of support points of $S$ is made.

1. Introduction. Let $H(\Delta)$ be the topological linear space of holomorphic functions on the unit disk $\Delta = \{z \in \mathbb{C}: |z| < 1\}$. The class $S$ is the subset of $H(\Delta)$ consisting of univalent functions $h$ with the normalization $h(0) = 0$, $h'(0) = 1$. We suppose $L$ is a complex-valued continuous linear functional on $H(\Delta)$, that is $L \in H(\Delta)^*$, and that $L$ is nonconstant on $S$. Then any function $f \in S$ for which $\Re L(f) = \max \{\Re L(h): h \in S\}$ must map $\Delta$ onto the complement of an analytic arc $\Gamma_f$ satisfying (except possibly at the finite endpoint)

$$\int (f^2/(f - w))(dw/w)^2 > 0 \quad (w \in \Gamma_f).$$

(See [8, 7, 4].) Such a function $f$ is called a support point of $S$ (corresponding to $L$).

In the most important examples—coefficient functionals, or more generally evaluation of a derivative of some order at some point of $\Delta$, or a linear combination of such functionals—$L(f^2/(f - w))$ is a rational function of $w$, and in a recent paper [5] P. L. Duren, Y. J. Leung, and M. M. Schiffer defined $L$ to be of rational type if $L(h^2/(h - w))$ is rational for all $h \in S$. This requirement appears to be excessively strong since only support points of $S$, in fact only support points which maximize $\Re L$ necessarily occur in (1). We show, however, in Theorem 1 that if $L(h^2/(h - w))$ is rational for a single $h \in S$, then it is rational for any $h \in S$, and in fact $L$ is simply a finite linear combination of functionals of the form $h \rightarrow h^{(n)}(z_0)$ $(n$ a nonnegative integer and $z_0 \in \Delta)$. The key to the proof is the following representation of $L$:

$$L(h) = \frac{1}{2\pi i} \int_\gamma h(z)F(z)dz \quad (h \in H(\Delta)),$$

where $\gamma(t) = re^{it}$ $(0 \leq t \leq 2\pi)$, $0 < r < 1$, $F$ is holomorphic for $|z| \geq r$, and $F(\infty) = 0$. The function $F$ is obtained from the well-known Toeplitz representation [9]

$$L(h) = \sum_0^\infty a_nb_n \quad (h(z) = \sum_0^\infty a_nz^n, \lim |b_n|^{1/n} < 1)$$

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372
by defining $F(z) = \sum_{n=0}^{\infty} b_n/z^{n+1}$. This function, subject to the conditions stated, is uniquely determined by $L$, but we can of course make use of slight analytic continuations of $F$ and deform the circular contour $\gamma$ into certain other closed curves without changing the value of the integral in (2). Also, we can, if we wish, drop the first two terms $b_0/z + b_1/z^2$ of the Laurent expansion of $F$ about $\infty$ without changing the corresponding support points. In fact, if $J$ is the functional corresponding as in (2) to the function $F(z) - b_0/z - b_1/z^2$, and if $h \in S$, then $L(h) = b_1 + J(h)$. In addition to the three equivalent formulations of "functional of rational type" already alluded to, Theorem 1 contains the fourth formulation: The function $F$ in (2) is rational. Thus we can restate the definition of "rational type" as follows.

**DEFINITION.** The functional $L \in H(\Delta)^*$ is of rational type if any one of the conditions (a), (b), (c), (d) of Theorem 1 holds.

We observe from (2) that for any $f \in S$,

$$L(f^2/(f-w)) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)^2}{f(z)-w} F(z) \, dz$$

for $w \in \mathbb{C} \setminus f(\Delta)$, while the last integral furnishes an analytic continuation of $L(f^2/(f-w))$ to the exterior of the closed Jordan curve $f \circ \gamma$. Our initial lemma provides a new relation between the functions $L(f^2/(f-w))$ and $F$ as follows. First we note that

$$f^2/(f-w) = f + w + w^2/(f-w),$$

and therefore that

$$L(f^2/(f-w)) = L(f) + wL(1) + w^2L(1/(f-w)).$$

The lemma then shows how $F$ can simply and with great symmetry be expressed in terms of $L(1/(f-w))$. Later we make use of the fact, exhibited by the last identity, that $L(f^2/(f-w))$ is rational exactly when $L(1/(f-w))$ is rational. We remark that the method of the lemma can easily be adapted to express $F(z) - b_0/z - b_1/z^2$ in terms of $L(f^2/(f-w))$.

Several recent papers on $S$ ([1, 2, 3, 6]) have dealt with truncation of the omitted arc $\Gamma_f$ of a support point $f$. If after such a truncation the resulting region is contracted so as to be of the form $g(\Delta)$ with $g \in S$, then $g$ is again a support point of $S$. Indeed, if $f$ maximizes $\text{Re} L$ over $S$, $L \in H(\Delta)^*$, a new functional $J$ can rather explicitly be constructed in terms of $L$ and $\Gamma_f$ such that $g$ maximizes $\text{Re} J$. (We shall be more explicit in §4 below.) In Theorem 2 we show that if $L$ is of rational type, then so is $J$. Thus, roughly speaking, functionals of rational type are preserved by arc truncation. In fact, we shall see that in a certain sense the exact form of the functional is preserved (see (6) and (7)).

### 2. The dual relationship between $L(1/(f-w))$ and $F(z)$.

**Lemma.** Let $f \in S$, $\gamma$ be a positively oriented closed Jordan curve in $\Delta$, $F$ holomorphic on and outside $\gamma$ with $F(\infty) = 0$. Let

$$G(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{f(z)-w} F(z) \, dz \quad (w \text{ outside } f \circ \gamma).$$
Then (after a slight analytic continuation of $G$)

\[ F(z) = \frac{1}{2\pi i} \int_{f^{-1}(w)} \frac{1}{f^{-1}(w) - z} G(w) \, dw \quad (z \text{ outside } \gamma). \]

**Proof.** Since $F$ is holomorphic on and outside $\gamma$ we can replace $\gamma$ by another positively oriented curve $\delta$, slightly inside $\gamma$, and rewrite $G$ (actually analytically continue $G$) as follows.

\[ G(w) = \frac{1}{2\pi i} \int_{f^{-1}(w)} \frac{1}{f^{-1}(w) - w} F(\zeta) \, d\zeta \quad (w \text{ outside } f \circ \delta). \]

Then

\[ \frac{1}{2\pi i} \int_{f^{-1}(w) - z} \frac{1}{f^{-1}(w) - z} G(w) \, dw \]

\[ = \frac{1}{2\pi i} \int_{f^{-1}(w) - w} \frac{1}{f^{-1}(w) - z} d\zeta. \]

Since $z$ lies outside $\gamma$, $1/(f^{-1}(w) - z)$ is holomorphic inside and on $f \circ \gamma$. Also, for any $\zeta$ on the contour $\delta$, $f(\zeta)$ is inside $f \circ \gamma$. Hence, by Cauchy’s integral formula, the value of the expression in square brackets is

\[ -\left. \frac{1}{f^{-1}(w) - z} \right|_{w=f(\zeta)} = \frac{1}{z - \zeta}. \]

Therefore

\[ \frac{1}{2\pi i} \int_{f^{-1}(w) - z} \frac{1}{f^{-1}(w) - z} G(w) \, dw = \frac{1}{2\pi i} \int_{f^{-1}(w) - w} \frac{1}{f^{-1}(w) - z} \, d\zeta = F(z) \]

as asserted. We remark that in a similar way (5) implies (4).

3. The equivalent formulations of “rational type”.

**Theorem 1.** Let $L \in H(\Delta)^*$ with corresponding function $F$ as in (2). Then the following statements are equivalent:

(a) There exists $f \in S$ such that $L(f^2/(f - w))$ is a rational function of $w$.

(b) The function $F$ is rational.

(c) $L$ is a finite linear combination of functionals of the form $h \to h^{(n)}(z_0)$, where $n$ is a nonnegative integer (order of the derivative) and $z_0 \in \Delta$. ($n$ and $z_0$ can vary from term to term.)

(d) The function $L(f^2/(f - w))$ is rational for every $f \in S$ (Duren, Leung, Schiffer).

**Proof.** Let $f$ be as in (a). Then as mentioned earlier $L(1/(f - w))$ is rational or, equivalently, the function $G$ in (4) is rational. Since the poles of $G$ lie inside $f \circ \gamma$, and since $G(\infty) = 0$, $G$ must be a linear combination of functions of the form $(w - w_0)^{-k}$, with $w_0$ inside $f \circ \gamma$ and $k$ a positive integer. Therefore it follows from (5) that to prove (b) we need only show that the function

\[ F_0(z) = \frac{1}{2\pi i} \int_{f^{-1}(w) - z} \frac{1}{(w - w_0)^k} \, dw \quad (z \text{ outside } \gamma) \]
FUNCTIONALS OF RATIONAL TYPE

375

is (the restriction of) a rational function. But for \( z \) outside \( \gamma \) Cauchy’s integral formula for the \( (k-1) \)st derivative gives

\[
F_0(z) = \frac{1}{(k-1)!} \left( \frac{d}{dw} \right)^{k-1} \left[ \frac{1}{f^{-1}(w) - z} \right]_{w=w_0}.
\]

The right side of this formula is clearly a rational function of \( z \), and so (a) implies (b).

The proof that (b) implies (c) is similar: The function \( F(z) \) in (2) is a linear combination of terms \( (z - z_0)^{-k} \) with \( z_0 \) inside \( \gamma \) and \( k \geq 1 \). Hence for \( h \in H(\Delta) \), \( L(h) \) is a linear combination of terms of the form

\[
\frac{1}{2\pi i} \int_{\delta} \frac{h(z)}{(z-z_0)^k} \, dz = \frac{h^{(k-1)}(z_0)}{(k-1)!}.
\]

To prove (c) implies (d) we need consider only a single functional of the form described in (c). The conclusion in (d) then clearly follows, and the proof of Theorem 1 is complete.

4. Truncation. As an application of Theorem 1 we discuss arc truncation of support points. Let \( f \) be a support point of \( S \), \( g \in S \), \( r > 1 \), and \( f < rg \). Geometrically, a portion of the omitted arc of \( f \) is removed and \( rg \) maps \( \Delta \) onto the complement of the remaining arc. Then it is now well known that \( g \) is also a support of \( S (1, 2, 3, 6) \). Briefly, if \( f \) maximizes \( \text{Re} L \) \( (L \in H(\Delta)^*) \), \( \varphi \) is defined by \( f = rg \circ \varphi = g \circ \varphi \circ \varphi'(0) \), and the functional \( J \) is defined by \( J(h) = L(h \circ \varphi) \), then \( J \in H(\Delta)^* \) and \( g \) maximizes \( \text{Re} J \) over \( S \). (One also shows that \( L \) nonconstant on \( S \) implies \( J \) nonconstant on \( S \).) We assert that if \( L \) is of rational type, then so is \( J \). In fact the following more general and, at the same time, more explicit theorem holds.

**Theorem 2.** Let \( L \) be of rational type, say

\[
L(h) = \sum_{k=1}^{K} \sum_{n=0}^{N_k} a_{nk} h^{(n)}(z_k) \quad (h \in H(\Delta)),
\]

where \( K \geq 1 \), \( N_K \geq 0 \) \( (1 \leq k \leq K) \), \( a_{nk} \in \mathbb{C} \) \( (1 \leq k \leq K, 0 \leq n \leq N_k) \), and \( z_k \in \Delta \) \( (1 \leq k \leq K) \). Then for any \( \varphi \in H(\Delta) \) with \( \varphi(\Delta) \subset \Delta \) there exist \( b_{nk} \in \mathbb{C} \) \( (1 \leq k \leq K, 0 \leq n \leq N_k) \) such that

\[
L(h \circ \varphi) = \sum_{k=1}^{K} \sum_{n=0}^{N_k} b_{nk} h^{(n)}(\varphi(z_k)) \quad (h \in H(\Delta)).
\]

**Proof.** By induction one sees that for \( n = 1, 2, 3, \ldots, \)

\[
(h \circ \varphi)^{(n)} = \sum_{j=1}^{n} (\psi_{jn})(h^{(j)} \circ \varphi),
\]

where \( \psi_{jn} \) is a function obtainable from \( \varphi \) by differentiation. (More explicitly:

\[
\psi_{jn} = \sum_{\nu_1 + \cdots + \nu_j = n} I_{\nu_1, \ldots, \nu_j} \varphi^{(\nu_1)} \cdots \varphi^{(\nu_j)}
\]

is a polynomial in \( \varphi \) of degree \( n \). By Theorem 1

\[
J(h) = L(h \circ \varphi) = \sum_{k=1}^{K} \sum_{n=0}^{N_k} b_{nk} h^{(n)}(\varphi(z_k))
\]

is a linear combination of terms of the form

\[
\frac{1}{2\pi i} \int_{\delta} \frac{h(z)}{(z-z_0)^k} \, dz = \frac{h^{(k-1)}(z_0)}{(k-1)!}.
\]
for certain positive integers $I_{\nu_1, \ldots, \nu_j}$.) Taking the trivial case $n = 0$ into account we can write

$$(h \circ \varphi)^{(n)} = \sum_{j=0}^{n} \left( \sum_{j=0}^{n} (\psi_jn)(h(j) \circ \varphi) \right) \quad (n = 0, 1, 2, \ldots),$$

where $\psi_0 = 0$ for $n > 0$ and $\psi_0 = 1$. Therefore, by (6),

$$L(h \circ \varphi) = \frac{1}{2\pi i} \int_{\gamma} \frac{F(\zeta)}{z - \varphi(\zeta)} \, d\zeta \quad (z \text{ in unbounded component of } \mathbb{C}\setminus\varphi \circ \gamma).$$

Theorem 2 applies to arc truncation only when $\varphi(0) = 0$. Then, in the “truncated functional” (7), all the derivatives $h^{(n)}$ are evaluated at points closer to the origin as compared with corresponding points in (6); that is, $|\varphi(z_k)| \leq |z_k|$. In particular, if $L$ is a linear combination of Maclaurin coefficients $(K = 1, z_1 = 0)$, then so is $J$. Of course this case is very special and can be seen without the theorem.

Finally, we remark that Cauchy’s integral formula provides an answer to the following question: If $L \in H(\Delta)^*$ with corresponding function $F$ as in (2), and if $\varphi \in H(\Delta)$ with $\varphi(\Delta) \subset \Delta$, then what function $H$ corresponds to the functional $h \to L(h \circ \varphi)$? The answer is:

$$H(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{F(\zeta)}{z - \varphi(\zeta)} \, d\zeta \quad (z \text{ in unbounded component of } \mathbb{C}\setminus\varphi \circ \gamma).$$

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