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FUNCTIONALS OF RATIONAL TYPE OVER THE CLASS S

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ABSTRACT. Let L be a continuous linear functional on the space of functions holomorphic in the unit disk, and let f be a function in the class S for which $\operatorname{Re} L$ achieves its maximum on S. Then L is said to be of rational type if the expression $L(f^2/(f-w))$, which occurs in Schiffer's differential equation, is a rational function of w. Various equivalent formulations of "rational type" are found and an application to the process of arc truncation of support points of S is made.

1. Introduction. Let $H(\Delta)$ be the topological linear space of holomorphic functions on the unit disk $\Delta = \{z \in \mathbf{C}\colon |z| < 1\}$. The class S is the subset of $H(\Delta)$ consisting of univalent functions h with the normalization $h(0) = 0$, $h'(0) = 1$. We suppose L is a complex-valued continuous linear functional on $H(\Delta)$, that is $L \in H(\Delta)^*$, and that L is nonconstant on S. Then any function $f \in S$ for which $\operatorname{Re} L(f) = \max \operatorname{Re}\{L(h)\colon h \in S\}$ must map Δ onto the complement of an analytic arc Γ_f satisfying (except possibly at the finite endpoint)

$$L(f^2/(f-w))(dw/w)^2 > 0 \qquad (w \in \Gamma_f). \tag{1}$$

(See [**8, 7, 4**].) Such a function f is called a support point of S (corresponding to L).

In the most important examples—coefficient functionals, or more generally evaluation of a derivative of some order at some point of Δ, or a linear combination of such functionals—$L(f^2/(f-w))$ is a rational function of w, and in a recent paper [**5**] P. L. Duren, Y. J. Leung, and M. M. Schiffer defined L to be of *rational type* if $L(h^2/(h-w))$ is rational for all $h \in S$. This requirement appears to be excessively strong since only support points of S, in fact only support points which maximize $\operatorname{Re} L$ necessarily occur in (1). We show, however, in Theorem 1 that if $L(h^2/(h-w))$ is rational for a single $h \in S$, then it is rational for any $h \in S$, and in fact L is simply a finite linear combination of functionals of the form $h \to h^{(n)}(z_0)$ (n a nonnegative integer and $z_0 \in \Delta$). The key to the proof is the following representation of L:

$$L(h) = \frac{1}{2\pi i}\int_\gamma h(z)F(z)\,dz \qquad (h \in H(\Delta)), \tag{2}$$

where $\gamma(t) = re^{it}$ $(0 \le t \le 2\pi)$, $0 < r < 1$, F is holomorphic for $|z| \ge r$, and $F(\infty) = 0$. The function F is obtained from the well-known Toeplitz representation [**9**]

$$L(h) = \sum_0^\infty a_n b_n \qquad \left(h(z) = \sum_0^\infty a_n z^n,\ \overline{\lim}\,|b_n|^{1/n} < 1\right)$$

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by defining $F(z) = \sum_0^\infty b_n/z^{n+1}$. This function, subject to the conditions stated, is uniquely determined by L, but we can of course make use of slight analytic continuations of F and deform the circular contour γ into certain other closed curves without changing the value of the integral in (2). Also, we can, if we wish, drop the first two terms $b_0/z + b_1/z^2$ of the Laurent expansion of F about ∞ without changing the corresponding support points. In fact, if J is the functional corresponding as in (2) to the function $F(z) - b_0/z - b_1/z^2$, and if $h \in S$, then $L(h) = b_1 + J(h)$. In addition to the three equivalent formulations of "functional of rational type" already alluded to, Theorem 1 contains the fourth formulation: The function F in (2) is rational. Thus we can restate the definition of "rational type" as follows.

DEFINITION. The functional $L \in H(\Delta)^*$ is of *rational type* if any one of the conditions (a), (b), (c), (d) of Theorem 1 holds.

We observe from (2) that for any $f \in S$,

$$L(f^2/(f-w)) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)^2}{f(z)-w}F(z)\,dz \tag{3}$$

for $w \in \mathbf{C} \setminus f(\Delta)$, while the last integral furnishes an analytic continuation of $L(f^2/(f-w))$ to the exterior of the closed Jordan curve $f \circ \gamma$. Our initial lemma provides a new relation between the functions $L(f^2/(f-w))$ and F as follows. First we note that

$$f^2/(f-w) = f + w + w^2/(f-w),$$

and therefore that

$$L(f^2/(f-w)) = L(f) + wL(1) + w^2L(1/(f-w)).$$

The lemma then shows how F can simply and with great symmetry be expressed in terms of $L(1/(f-w))$. Later we make use of the fact, exhibited by the last identity, that $L(f^2/(f-w))$ is rational exactly when $L(1/(f-w))$ is rational. We remark that the method of the lemma can easily be adapted to express $F(z) - b_0/z - b_1/z^2$ in terms of $L(f^2/(f-w))$.

Several recent papers on S ([**1, 2, 3, 6**]) have dealt with truncation of the omitted arc Γ_f of a support point f. If after such a truncation the resulting region is contracted so as to be of the form $g(\Delta)$ with $g \in S$, then g is again a support point of S. Indeed, if f maximizes $\operatorname{Re} L$ over S, $L \in H(\Delta)^*$, a new functional J can rather explicitly be constructed in terms of L and Γ_f such that g maximizes $\operatorname{Re} J$. (We shall be more explicit in §4 below.) In Theorem 2 we show that if L is of rational type, then so is J. Thus, roughly speaking, functionals of rational type are preserved by arc truncation. In fact, we shall see that in a certain sense the exact form of the functional is preserved (see (6) and (7)).

2. The dual relationship between $L(1/(f-w))$ and $F(z)$.

LEMMA. *Let $f \in S$, γ be a positively oriented closed Jordan curve in Δ, F holomorphic on and outside γ with $F(\infty) = 0$. Let*

$$G(w) = \frac{1}{2\pi i}\int_\gamma \frac{1}{f(z)-w}F(z)\,dz \qquad (w \text{ outside } f\circ\gamma). \tag{4}$$

Then (after a slight analytic continuation of G)

$$F(z) = \frac{1}{2\pi i}\int_{f\circ\gamma} \frac{1}{f^{-1}(w)-z} G(w)\,dw \qquad (z \text{ outside } \gamma). \tag{5}$$

PROOF. Since F is holomorphic on and outside γ we can replace γ by another positively oriented curve δ, slightly inside γ, and rewrite G (actually analytically continue G) as follows.

$$G(w) = \frac{1}{2\pi i}\int_{\delta} \frac{1}{f(\varsigma)-w} F(\varsigma)\,d\varsigma \qquad (w \text{ outside } f\circ\delta).$$

Then

$$\frac{1}{2\pi i}\int_{f\circ\gamma} \frac{1}{f^{-1}(w)-z} G(w)\,dw$$
$$= \frac{1}{2\pi i}\int_{\delta}\left[\frac{1}{2\pi i}\int_{f\circ\gamma} \frac{1}{f(\varsigma)-w}\frac{1}{f^{-1}(w)-z}\,dw\right] F(\varsigma)\,d\varsigma.$$

Since z lies outside γ, $1/(f^{-1}(w)-z)$ is holomorphic inside and on $f\circ\gamma$. Also, for any ς on the contour δ, $f(\varsigma)$ is inside $f\circ\gamma$. Hence, by Cauchy's integral formula, the value of the expression in square brackets is

$$-\frac{1}{f^{-1}(w)-z}\bigg|_{w=f(\varsigma)} = \frac{1}{z-\varsigma}.$$

Therefore

$$\frac{1}{2\pi i}\int_{f\circ\gamma} \frac{1}{f^{-1}(w)-z} G(w)\,dw = \frac{1}{2\pi i}\int_{\delta} \frac{F(\varsigma)}{z-\varsigma}\,d\varsigma = F(z)$$

as asserted. We remark that in a similar way (5) implies (4).

3. The equivalent formulations of "rational type".

THEOREM 1. *Let $L \in H(\Delta)^*$ with corresponding function F as in* (2). *Then the following statements are equivalent:*

(a) *There exists $f \in S$ such that $L(f^2/(f-w))$ is a rational function of w.*

(b) *The function F is rational.*

(c) *L is a finite linear combination of functionals of the form $h \to h^{(n)}(z_0)$, where n is a nonnegative integer (order of the derivative) and $z_0 \in \Delta$. (n and z_0 can vary from term to term.)*

(d) *The function $L(f^2/(f-w))$ is rational for every $f \in S$ (Duren, Leung, Schiffer).*

PROOF. Let f be as in (a). Then as mentioned earlier $L(1/(f-w))$ is rational or, equivalently, the function G in (4) is rational. Since the poles of G lie inside $f\circ\gamma$, and since $G(\infty)=0$, G must be a linear combination of functions of the form $(w-w_0)^{-k}$, with w_0 inside $f\circ\gamma$ and k a positive integer. Therefore it follows from (5) that to prove (b) we need only show that the function

$$F_0(z) = \frac{1}{2\pi i}\int_{f\circ\gamma} \frac{1}{f^{-1}(w)-z}\frac{1}{(w-w_0)^k}\,dw \qquad (z \text{ outside } \gamma)$$

is (the restriction of) a rational function. But for z outside γ Cauchy's integral formula for the $(k-1)$st derivative gives

$$F_0(z) = \frac{1}{(k-1)!}\left(\frac{d}{dw}\right)^{k-1}\left[\frac{1}{f^{-1}(w)-z}\right]_{w=w_0}.$$

The right side of this formula is clearly a rational function of z, and so (a) implies (b).

The proof that (b) implies (c) is similar: The function $F(z)$ in (2) is a linear combination of terms $(z-z_0)^{-k}$ with z_0 inside γ and $k \geq 1$. Hence for $h \in H(\Delta)$, $L(h)$ is a linear combination of terms of the form

$$\frac{1}{2\pi i}\int_\delta \frac{h(z)}{(z-z_0)^k}\,dz = \frac{h^{(k-1)}(z_0)}{(k-1)!}.$$

To prove (c) implies (d) we need consider only a single functional of the form described in (c). The conclusion in (d) then clearly follows, and the proof of Theorem 1 is complete.

4. Truncation. As an application of Theorem 1 we discuss arc truncation of support points. Let f be a support point of S, $g \in S$, $r > 1$, and $f \prec rg$. Geometrically, a portion of the omitted arc of f is removed and rg maps Δ onto the complement of the remaining arc. Then it is now well known that g is also a support of S ([**1, 2, 3, 6**]). Briefly, if f maximizes $\operatorname{Re} L$ ($L \in H(\Delta)^*$), φ is defined by $f = rg\circ\varphi = g\circ\varphi/\varphi'(0)$, and the functional J is defined by $J(h) = L(h\circ\varphi)$, then $J \in H(\Delta)^*$ and g maximizes $\operatorname{Re} J$ over S. (One also shows that L nonconstant on S implies J nonconstant on S.) We assert that if L is of rational type, then so is J. In fact the following more general and, at the same time, more explicit theorem holds.

THEOREM 2. *Let L be of rational type, say*

$$L(h) = \sum_{k=1}^{K}\sum_{n=0}^{N_k} a_{nk}h^{(n)}(z_k) \qquad (h \in H(\Delta)), \tag{6}$$

where $K \geq 1$, $N_K \geq 0$ $(1 \leq k \leq K)$, $a_{nk} \in \mathbf{C}$ $(1 \leq k \leq K,\ 0 \leq n \leq N_k)$, and $z_k \in \Delta$ $(1 \leq k \leq K)$. Then for any $\varphi \in H(\Delta)$ with $\varphi(\Delta) \subset \Delta$ there exist $b_{nk} \in \mathbf{C}$ $(1 \leq k \leq K,\ 0 \leq n \leq N_k)$ such that

$$L(h\circ\varphi) = \sum_{k=1}^{K}\sum_{n=0}^{N_k} b_{nk}h^{(n)}(\varphi(z_k)) \qquad (h \in H(\Delta)). \tag{7}$$

PROOF. By induction one sees that for $n = 1, 2, 3, \ldots$,

$$(h\circ\varphi)^{(n)} = \sum_{j=1}^{n}(\psi_{jn})(h^{(j)}\circ\varphi),$$

where ψ_{jn} is a function obtainable from φ by differentiation. (More explicitly:

$$\psi_{jn} = \sum_{\nu_1+\cdots+\nu_j=n} I_{\nu_1,\ldots,\nu_j}\varphi^{(\nu_1)}\cdots\varphi^{(\nu_j)}$$

for certain positive integers $I_{\nu_1,\ldots,\nu_j}$.) Taking the trivial case $n = 0$ into account we can write

$$(h \circ \varphi)^{(n)} = \sum_{j=0}^{n} (\psi_{jn})(h^{(j)} \circ \varphi) \qquad (n = 0, 1, 2, \ldots),$$

where $\psi_{0n} = 0$ for $n > 0$ and $\psi_{00} = 1$. Therefore, by (6),

$$\begin{aligned} L(h \circ \varphi) &= \sum_{k=1}^{K} \sum_{n=0}^{N_k} a_{nk} \left[\sum_{j=0}^{n} \psi_{jn}(z_k) h^{(j)}(\varphi(z_k)) \right] \\ &= \sum_{k=1}^{K} \sum_{j=0}^{N_k} \left[\sum_{n=j}^{N_k} a_{nk} \psi_{jn}(z_k) \right] h^{(j)}(\varphi(z_k)) \end{aligned}$$

as required.

Theorem 2 applies to arc truncation only when $\varphi(0) = 0$. Then, in the "truncated functional" (7), all the derivatives $h^{(n)}$ are evaluated at points closer to the origin as compared with corresponding points in (6); that is, $|\varphi(z_k)| \leq |z_k|$. In particular, if L is a linear combination of Maclaurin coefficients ($K = 1,\ z_1 = 0$), then so is J. Of course this case is very special and can be seen without the theorem.

Finally, we remark that Cauchy's integral formula provides an answer to the following question: If $L \in H(\Delta)^*$ with corresponding function F as in (2), and if $\varphi \in H(\Delta)$ with $\varphi(\Delta) \subset \Delta$, then what function H corresponds to the functional $h \to L(h \circ \varphi)$? The answer is:

$$H(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{F(\varsigma)}{z - \varphi(\varsigma)}\, d\varsigma \qquad (z \text{ in unbounded component of } \mathbf{C} \backslash \varphi \circ \gamma).$$

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References

1. L. Brickman and P. L. Duren, *Truncation of support points for univalent functions*, Complex Variables Theory Appl. **3** (1984), 71–83.
2. L. Brickman and Y. J. Leung, *Exposed points of the set of univalent functions*, Bull. London Math. Soc. **16** (1984), 157–159.
3. L. Brickman, Y. J. Leung and D. R. Wilken, *On extreme points and support points of the class S*, Ann. Univ. Mariae Curie-Sklodowska (Krzyz special issue) (to appear).
4. L. Brickman and D. R. Wilken, *Support points of the set of univalent functions*, Proc. Amer. Math. Soc. **42** (1974), 523–528.
5. P. L. Duren, Y. J. Leung and M. M. Schiffer, *Support points with maximum radian angle*, Complex Variables Theory Appl. **1** (1983), 263–277.
6. W. E. Kirwan and G. Schober, *New inequalities from old ones*, Math. Z. **180** (1982), 19–40.
7. A. Pfluger, *Lineare Extremalprobleme bei schlichten Funktionen*, Ann. Acad. Sci. Fenn. Ser. AI **489** (1971).
8. M. M. Schiffer, *A method of variation within the family of simple functions*, Proc. London Math. Soc. (2) **44** (1938), 432–449.
9. O. Toeplitz, *Die linearen vollkommenen Räume der Funktionentheorie*, Comment. Math. Helv. **23** (1949), 222–242.

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