A RESONANCE PROBLEM IN WHICH THE NONLINEARITY MAY GROW LINEARLY

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ABSTRACT. The purpose of this paper is to study a semilinear two point boundary value problem of resonance type in which the nonlinear perturbation may grow linearly. A significant improvement of a recent result due to Cesari and Kannan is given.

We consider the boundary value problem

(1) $u'' + u + g(u) = h(x), \quad u(0) = u(\pi) = 0,$

where $h \in L^2[0, \pi]$ and $g$ is continuous. If the numbers

$$g(\infty) = \liminf_{\xi \to \infty} g(\xi) \quad \text{and} \quad g(-\infty) = \limsup_{\xi \to -\infty} g(\xi)$$

are finite and

(2) $g(-\infty) \int_0^\pi \sin x \, dx < \int_0^\pi h(x) \sin x \, dx < g(\infty) \int_0^\pi \sin x \, dx,$

then it follows from a slight variation of a well-known theorem due to Landesman and Lazer [3] that (1) has a solution. If the limits of $g$ at $\pm \infty$ exist and $g(-\infty) < g < g(\infty)$, these conditions are also necessary for the solvability of (1). We show that if $g$ satisfies an additional restriction, then (2) implies the existence of a solution of (1) even if one or both of the numbers $g(\infty)$ and $g(-\infty)$ is infinite. As will be shown in a remark after the proof, the following result is a strong improvement of the main result in [1].

For the case in which the inequalities (2) are reversed, see, for example, [4]. Other solvability conditions in the absence of the Landesman-Lazer conditions are given in [5 and 6].

THEOREM. If there exist numbers $\gamma$ and $r$, $r > 0$, with $0 < \gamma < 3$ such that

(3) $g(\xi) / \xi \leq \gamma, \quad |\xi| \geq r$

and (2) holds, then (1) is solvable.

(By a solution we mean a function with absolutely continuous derivative which satisfies the boundary conditions and the differential equation a.e.)

PROOF. To prove the theorem we use the well-known continuation method of Leray and Schauder.
For each $s$ in $[0,1]$ we consider the boundary value problem
\begin{equation}
\tag{4}
u'' + (1 + \gamma)u + s(g(u) - \gamma u) = h(x), \quad u(0) = u(\pi) = 0.
\end{equation}

For $v$ in $C[0,\pi]$, let $|v|_\infty = \max_{[0,\pi]} |v(t)|$. We claim that there exists a number $R$ independent of $s$, $s \in [0,1]$, such that if $u$ is a solution of (4), then $|u|_\infty < R$.

In order to establish this claim, we first consider a certain decomposition of the function $g$. Let $\Phi$ be a smooth function such that $\Phi(\xi) = 0$ if $|\xi| < \tau$, $0 < \Phi(\xi) < 1$ for all $\xi$, and $\Phi(\xi) = 1$ if $|\xi| \geq 2\tau$. If we set $g_1(\xi) = \Phi(\xi)g(\xi)$ and $g_2(\xi) = (1 - \Phi(\xi))g(\xi)$, then $g_2(\xi)$ is bounded on $(-\infty, \infty)$, and (2) and (3) imply the existence of a number $m$ such that
\begin{equation}
\tag{5}m \leq g_1(\xi)/\xi \leq \gamma
\end{equation}
for $\xi \neq 0$. Setting $g(\xi)/\xi$ equal to 0 when $\xi = 0$, we may assume that (5) holds for all $\xi$, $\xi \in (-\infty, \infty)$.

If we assume that the claim is false, then there exists a sequence of numbers $\{s_n\}_{n=1}^\infty$ in $[0,1]$ and a corresponding sequence of functions $\{u_n\}_{n=1}^\infty$ such that $u_n$ is a solution of (4) when $s = s_n$ and $|u_n|_\infty \geq n$ for all $n$. If we set $v_n = u_n/|u_n|_\infty$ for all $n$, then
\begin{equation}
\tag{6}v_n'' + v_n + p_n(x)v_n = h_n(x), \quad v_n(0) = v_n(\pi) = 0,
\end{equation}
where
\begin{align*}
p_n(x) &= (1 - s_n)\gamma + s_n g_1(u_n(x))/u_n(x), \\
h_n(x) &= [h(x) - s_n g_2(u_n(x))]/|u_n|_\infty.
\end{align*}

From (5) we infer the existence of a number $m_1$ such that
\begin{equation}
\tag{9}m_1 \leq p_n(x) \leq \gamma
\end{equation}
for all $x$. Moreover, since $g_2$ is bounded, it follows that $\lim_{n \to \infty} h_n = 0$ in $L^2[0,\pi]$. Since $|v_n|_\infty = 1$ for all $n$, it follows that the $L^2[0,\pi]$ norm of $v_n''$ is bounded independently of $n$. Since, by Rolle's theorem, $v'_n$ vanishes somewhere on $(0,\pi)$, we see that the sequence $\{v'_n\}_{n=1}^\infty$ is equicontinuous and uniformly bounded on $[0,\pi]$. Therefore, the sequence $\{v_n\}_{n=1}^\infty$ is also equicontinuous and uniformly bounded on $[0,\pi]$. Hence, by Ascoli's lemma we may assume that $\lim_{n \to \infty} v_n(x) = w(x)$, and $\lim v'_n(x) = w'(x)$ uniformly on $[0,\pi]$, where $w \in C^1[0,\pi]$, $|w|_\infty = 1$, and $w(0) = w(\pi) = 0$. By (9), the sequence $\{p_n\}_{n=1}^\infty$ is bounded in $L^2[0,\pi]$. Hence we may assume that $p_n$ converges weakly to a function $p(x)$ in $L^2[0,\pi]$.

It follows that $m_1 \leq p(x) \leq \gamma$ a.e. on $[0,\pi]$. This follows from (9), since, by Mazur's theorem, closed subsets of $L^2[0,\pi]$ are weakly closed.

From (6) we see that for $x$ in $[0,\pi]$
\begin{equation}
\tag{10}v'_n(x) = v'_n(0) - \int_0^x (1 + p_n(t)v_n(t)) \, dt - \int_0^x h_n(t) \, dt.
\end{equation}

It follows, by letting $n \to \infty$ in (10), that
\begin{equation}
\tag{11}w'(x) = w'(0) - \int_0^x (1 + p(x))w(x) \, dx.
\end{equation}

Therefore, $w'$ is absolutely continuous and
\begin{equation}
\tag{12}w''(x) + (1 + p(x))w(x) = 0 \quad \text{a.e.,} \quad w(0) = w(\pi) = 0.
\end{equation}
We assert that \( w(x) \neq 0 \) for all \( x \) in \( (0, \pi) \). Indeed, since \( w(x) \neq 0 \) and \( 1 + p(x) \leq 1 + \gamma < 4 \) a.e., if \( w(\xi) = 0 \) for some \( \xi \) in \( (0, \pi) \), then, by the Sturm Comparison theorem, every nontrivial solution of \( y'' + Ay = 0 \) will have to vanish on each of the intervals \((0, \xi)\) and \((\xi, \pi)\). Since \( \sin 2t \) has one zero on \((0, \pi)\), the assertion follows.

We will assume that \( w(x) > 0 \) for all \( x \) in \((0, \pi)\), and arrive at a contradiction. The alternative \( w(x) < 0 \) on \((0, \pi)\) will also lead to a contradiction.

Since \( w \) is a nontrivial solution of (11), \( w'(0) > 0 \) and \( w'(\pi) < 0 \). Since

\[
\frac{u_n(x)}{|u_n(\infty)|} = w(x)
\]

as \( n \to \infty \) in the norm of \( C^1[0, \pi] \), it follows that \( u_n(x) > 0 \) on \([0, \pi]\) for large \( n \). Consequently (2) implies that the function

\[
(1 \cdot s_n) \gamma u_n(x) + s_n g(u_n(x)) = z_n(x)
\]

is bounded below on \([0, \pi]\) independently of \( n \). Since \( |u_n(\infty)| \to \infty \) as \( n \to \infty \) and \( w(x) > 0 \) on \((0, \pi)\), it follows that \( u_n(x) \to \infty \) uniformly on compact subintervals of \((0, \pi)\), and hence, by (2), we have

\[
\int_0^\pi h(x) \sin x \, dx < \int_0^\pi \left( \liminf_{n \to \infty} z_n(x) \right) \sin x \, dx.
\]

Multiplying the differential equation in (4) by \( \sin x \) when \( s = s_n \) and \( u = u_n \) and integrating by parts, we obtain

\[
\int_0^\pi z_n(x) \sin x \, dx = \int_0^\pi h(x) \sin x \, dx.
\]

Since \( z_n(x) \) is bounded below on \([0, \pi]\) independently of \( n \), Fatou’s lemma implies that

\[
\int_0^\pi \left( \liminf_{n \to \infty} z_n(x) \right) \sin x \, dx \leq \liminf_{n \to \infty} \int_0^\pi z_n(x) \sin x \, dx = \int_0^\pi h(x) \sin x \, dx,
\]

which contradicts (12). Therefore, the existence of the a priori bound is established.

The proof now follows more or less along standard lines. Since the problem

\[
Lu = u'' + u + \gamma u = 0, \quad u(0) = u(\pi) = 0,
\]

has no solution other than \( u \equiv 0 \), for any \( f \in C[0, 1] \) the problem

\[
Lu = f, \quad u(0) = u(\pi) = 0,
\]

has a unique solution, which we denote by \( L^{-1} f \). By standard results, \( L^{-1} \) may be considered as a compact mapping from the Banach space \( C[0, \pi] \) into itself. Let \( G: C[0, \pi] \to C[0, \pi] \) be the Nemytskii map associated with \( g \) and let \( u_0 \) be the unique solution satisfying

\[
Lu_0 = h, \quad u_0(0) = u_0(\pi) = 0.
\]

Since \( G \) is continuous and takes bounded subsets of \( C[0, \pi] \) into bounded subsets of \( C[0, \pi] \), the mapping \( N: C[0, \pi] \times [0, 1] \to C[0, 1] \) given by

\[
N(u, s) = u_0 + L^{-1}[s(\gamma u - G(u))]
\]

is a compact homotopy. If \( u = N(u, s) \) for some \( s \) in \([0, 1]\) then \( u \) is a solution of (4). Hence it follows from what has been shown above that there exists a number \( R \),
$R > 0$, such that $|u|_\infty < R$. Let $D = \{u \in C[0, \pi] | |u|_\infty < R\}$. Since $u - N(u, s) \neq 0$ for all $(u, s) \in \partial D \times [0, 1]$, it follows from the homotopy invariance theorem of degree theory (see, for example, [2]) that the Leray-Schauder degree $d(I - N(\cdot, s), D, 0)$ is constant for $s \in [0, 1]$. Since $N(u, 0) = u_0$ and our previous argument, applied to (4) with $s = 0$, shows that $|u_0| < R$, it follows that

$$1 = d(I - N(\cdot, 0), D, 0) = d(I - N(\cdot, 1), D, 0).$$

Hence $u = N(u, 1)$ has a solution which is a solution of (1). This proves the theorem.

**Remark.** Cesari and Kannan [1] consider problem (1) under the assumption that $g$ is nondecreasing (actually, they write the differential equation in the form $u'' + u - g(u) = h$ and assume $g$ to be nonincreasing). They assume that there exists a constant $\gamma$ with $0 < \gamma < 0.443$ such that $|g(\xi)| < c + \gamma|\xi|$ for some $c, c > 0$. They further assume that

$$\limsup_{\xi \to -\infty} g(\xi)/\xi = \gamma = -\liminf_{\xi \to -\infty} g(\xi)/\xi.$$

Obviously these assumptions imply that $g(\infty) = \infty$ and $g(-\infty) = -\infty$, so that (2) holds trivially. Also their assumptions imply (3) (with a different $\gamma, \gamma < 3$). Hence their result is a special case of our theorem. We emphasize that in Theorem 1 it is not necessary that $g$ be monotone. Moreover, since the problem

$$u'' + u + 3u = \sin 2t \quad u(0) = u(\pi) = 0,$$

has no solution, the condition $\gamma < 3$ is sharp. Cesari and Kannan’s condition that $\gamma < 0.443$ improves the condition $\gamma < 0.24347$ obtained earlier by Schechter, Shapiro and Snow in [7]. In [7] it was assumed that $g$ is odd as well as nondecreasing.

**References**