

A FAMILY OF POLYNOMIALS WITH CONCYCLIC ZEROS. II

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ABSTRACT. Let $\lambda_1, \dots, \lambda_J$ be nonzero real numbers. Expand

$$E(z) = \prod (-1 + \exp \lambda_j z),$$

rewrite products of exponentials as single exponentials, and replace every $\exp(az)$ by its approximation $(1 + an^{-1}z)^n$, where $n \geq J$. The resulting polynomial has all zeros on the (possibly infinite) circle of radius $|r|$ centered at $-r$, where $r = n/\sum \lambda_j$.

1. Introduction. Our purpose is to establish Conjecture [1] of [S2]. For positive integers n let P_n be the linear mapping from the exponential polynomials over \mathbb{C} to the polynomials over \mathbb{C} that replaces $\exp(az)$ by

$$(1.1) \quad \left(1 + \frac{az}{n}\right)^n$$

but is otherwise the identity. For example,

$$P_n\{(e^{5z} - 1)(e^z - 1)\} = \left(1 + \frac{6z}{n}\right)^n - \left(1 + \frac{5z}{n}\right)^n - \left(1 + \frac{z}{n}\right)^n + 1.$$

Thus P_∞ applied to any exponential polynomial $E(z)$ would be the identity. Next, a set of points in the complex plane is said to be concyclic if each of its points lies on the same circle, or on the same line.

The above-mentioned conjecture is now the

THEOREM. Assume $n \geq J$. Let the λ_j , for $1 \leq j \leq J$, be nonzero real numbers. Then the zeros of $P_n E(z)$, where

$$(1.2) \quad E(z) = \prod_{j=1}^J (e^{\lambda_j z} - 1),$$

are concyclic. In fact, they all lie on $C(r)$, the circle of radius $|r|$ centered at $-r$, where

$$(1.3) \quad r = n / \sum_{j=1}^J \lambda_j.$$

If $\sum \lambda_j = 0$, this means the zeros are purely imaginary.

The condition $n \geq J$ is needed to insure that $P_n E(z)$ is not identically zero. The fact that $P_n E(z)$ is identically zero if and only if $J > n$ is established in the course of the proof (see formula (3.5)).

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Our proof uses a theorem of N. Obrechhoff [O] and seems quite different from the approach used in [S1] where a theorem of A. Cohn [C] was used to obtain partial results. However, Cohn’s theorem can be used to obtain a “ q -analogue” of these [S3].

We remark that the present method of proof also establishes the result for

$$(1.4) \quad E(z) = \sum_{j=1}^J (e^{\lambda_j z + i b_j} - 1)$$

where b_1, \dots, b_J are any real numbers. For other results related to zeros of exponential polynomials see [DeB, I, L-S] and the references of [S1].

2. A theorem of Obrechhoff. For complex α and a fixed real h let $T(\alpha) = T_h(\alpha)$ be the operator on the set of all polynomials that is defined by

$$(2.1) \quad T(\alpha)g(z) = g(z + h) + \alpha g(z - h).$$

In [O, pp. 95–97] Obrechhoff showed (his angular parameter ϕ may be set equal to zero with no loss of generality) that if α lies on the unit circle U (i.e. $|\alpha| = 1$) and the zeros of $g(z)$ lie in a vertical strip S , then the zeros of $T(\alpha)g(z)$ lie in the same strip S .

Now define an operator Δ_i by

$$(2.2) \quad \Delta_i g(z) = g(z + \lambda_i) - g(z).$$

LEMMA. *If all zeros of the nonconstant polynomial $g(z)$ lie on $\text{Re } z = \sigma$, then all zeros of $\Delta_i g(z)$ lie on*

$$(2.3) \quad \text{Re } z = \sigma - (\lambda_i/2).$$

PROOF. Let $s = z + (\lambda_i/2)$. Then

$$(2.4) \quad \Delta_i g(z) = g(s + (\lambda_i/2)) - g(s - (\lambda_i/2)).$$

Since all zeros of $g(s)$ have real part σ , the result follows from the case $\alpha = -1$ of Obrechhoff’s theorem.

COROLLARY. *All zeros of*

$$(2.5) \quad \Delta_1 \Delta_2 \cdots \Delta_J z^n = 0$$

lie on

$$(2.6) \quad \text{Re } z = - \left(\sum_{j=1}^J \lambda_j \right) / 2.$$

PROOF. Apply the above lemma J times with $\sigma = 0$.

The above lemma can also be deduced from a lemma in [T, p. 238].

3. Proof of the theorem. Clearly

$$(3.1) \quad R_n(z) := P_n E(z) = 1 - \sum_j \left(1 + \frac{\lambda_j z}{n} \right)^n + \sum_{i < j} \left(1 + \frac{(\lambda_i + \lambda_j)z}{n} \right)^n - \dots$$

Set $w = n/z$. Thus

$$(3.2) \quad w^n R_n(n/w) = w^n - \sum_j (w + \lambda_j)^n + \sum_{i < j} (w + \lambda_i + \lambda_j)^n - \dots$$

where the signs alternate and (consider w^n as the 0th sum on the right) the k th sum has the form

$$(3.3) \quad \sum (w + \lambda(j_1) + \dots + \lambda(j_k))^n$$

where $\lambda(j) = \lambda_j$ and the sum is over all k -tuples

$$(3.4) \quad j_1 < \dots < j_k.$$

It is now clear that

$$(3.5) \quad w^n R_n(n/w) = \Delta_1 \Delta_2 \dots \Delta_J w^n,$$

and that the right-hand side is not identically zero unless $J > n$. By the previous corollary,

$$(3.6) \quad R_n(z) = 0$$

implies

$$(3.7) \quad \operatorname{Re} w = - \left(\sum \lambda_j \right) / 2$$

so $z = n/w$ lies on a circle through the origin that is symmetric with respect to the real axis, and cuts the real axis at

$$(3.8) \quad x_0 = -2n / \sum \lambda_j.$$

Hence z must lie on a circle of radius $|r|$ and center $-r$ where

$$(3.9) \quad r = n / \sum_{j=1}^J \lambda_j.$$

If r is infinite, the above argument shows that the zeros all lie on the imaginary axis. This completes the proof.

REMARK. Define the operators Δ and B_J by

$$(3.10) \quad \Delta F(n) = F(n + 1) - F(n)$$

and

$$(3.11) \quad B_J F(n) = J!^{-1} \Delta^J F(n)$$

for any number theoretic function $F = F(n)$. Thus,

$$(3.12) \quad B_J F(n) = \frac{1}{J!} \sum_{k=0}^J (-1)^{J-k} \binom{J}{k} F(n + k).$$

If $F = F_u(n)$ is the function n^u , where u is a nonnegative integer, then it is well known that $B_J F_u(0)$ is a Stirling number of the second kind, and

$$(3.13) \quad B_J F_u(0) = \delta(J, u), \quad 0 \leq u \leq J,$$

where $\delta(i, j)$ is the Kronecker delta. Professor Graydon Bell of the University of Northern Arizona has pointed out to us that

$$(3.14) \quad B_J P_n e^z = \sum_{k=0}^J \frac{z^k}{k!}.$$

Hence the operator B_J provides a link between the two most common approximations to e^z . Formula (3.14) can be deduced from (3.13). Also, it can be inverted to yield

$$(3.15) \quad \left(1 + \frac{z}{n}\right)^n = \sum_{k=0}^n k n^{-k-1} \frac{n!}{(n-k)!} \sum_{j=0}^k \frac{z^j}{j!}.$$

REFERENCES

- [C] A. Cohn, *Über die Anzahl der Wurzeln einer algebraischen Gleichung in einem Kreise*, Math. Z. **14** (1922), 110–148.
- [DeB] N. G. de Bruijn, *The roots of trigonometric integrals*, Duke Math. J. **17** (1950), 197–226.
- [I] L. Ilieff, *Über die Nullstellen einiger Klassen von Polynomen*, Tôhoku Math. J. **45** (1939), 259–264.
- [L-S] E. H. Lieb and A. D. Sokal, *A general Lee-Yang theorem for one-component and multicomponent ferromagnets*, Comm. Math. Phys. **80** (1981), 153–179.
- [O] N. Obrechhoff, *Sur les racines des équations algébriques*, Tôhoku Math. J. **38** (1933), 93–100.
- [S1] K. B. Stolarsky, *A family of polynomials with concyclic zeros*, Proc. Amer. Math. Soc. **88** (1983), 622–624.
- [S2] —, *Zeros of exponential polynomials and “reductionism”*, Topics in Classical Number Theory, Colloq. Math. Soc. János Bolyai, vol. 34, Elsevier, North-Holland (to appear).
- [S3] —, *A family of polynomials with concyclic zeros*. III (in preparation).
- [T] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Clarendon Press, Oxford, 1951.

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