A FAMILY OF POLYNOMIALS WITH CONCYCLIC ZEROS. II
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ABSTRACT. Let $\lambda_1, \ldots, \lambda_J$ be nonzero real numbers. Expand
$$E(z) = \prod (-1 + \exp \lambda_j z),$$
rewrite products of exponentials as single exponentials, and replace every
$\exp(az)$ by its approximation $(1 + an^{-1}z)^n$, where $n \geq J$. The resulting
polynomial has all zeros on the (possibly infinite) circle of radius $|r|$ centered
at $-r$, where $r = n/\sum \lambda_j$.

1. Introduction. Our purpose is to establish Conjecture [1] of [S2]. For
positive integers $n$ let $P_n$ be the linear mapping from the exponential polynomials
over $\mathbb{C}$ to the polynomials over $\mathbb{C}$ that replaces $\exp(az)$ by
$$E(z) = \prod \left[ 1 + \frac{az}{n} \right]^n,$$
but is otherwise the identity. For example,
$$P_n \{(e^{5z} - 1)(e^z - 1)\} = \left(1 + \frac{6z}{n}\right)^n - \left(1 + \frac{5z}{n}\right)^n - \left(1 + \frac{z}{n}\right)^n + 1.$$ 
Thus $P_\infty$ applied to any exponential polynomial $E(z)$ would be the identity. Next,
a set of points in the complex plane is said to be concyclic if each of its points lies
on the same circle, or on the same line.

The above-mentioned conjecture is now the

THEOREM. Assume $n \geq J$. Let the $\lambda_j$, for $1 \leq j \leq J$, be nonzero real numbers.
Then the zeros of $P_n E(z)$, where
$$E(z) = \prod_{j=1}^J (e^{\lambda_j z} - 1),$$
are concyclic. In fact, they all lie on $C(r)$, the circle of radius $|r|$ centered at $-r$, where
$$r = n/\sum_{j=1}^J \lambda_j.$$ 
If $\sum \lambda_j = 0$, this means the zeros are purely imaginary.

The condition $n \geq J$ is needed to insure that $P_n E(z)$ is not identically zero. The
fact that $P_n E(z)$ is identically zero if and only if $J > n$ is established in the course
of the proof (see formula (3.5)).

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Our proof uses a theorem of N. Obrechkoff [O] and seems quite different from the approach used in [S1] where a theorem of A. Cohn [C] was used to obtain partial results. However, Cohn’s theorem can be used to obtain a “q-analogue” of these [S3].

We remark that the present method of proof also establishes the result for

\[
E(z) = \sum_{j=1}^{J} (e^{\lambda_j z + ib_j} - 1)
\]

where \(b_1, \ldots, b_J\) are any real numbers. For other results related to zeros of exponential polynomials see [DeB, I, L-S] and the references of [S1].

2. A theorem of Obrechkoff. For complex \(\alpha\) and a fixed real \(h\) let \(T(\alpha) = T_h(\alpha)\) be the operator on the set of all polynomials that is defined by

\[
T(\alpha)g(z) = g(z + h) + \alpha g(z - h).
\]

In [O, pp. 95–97] Obrechkoff showed (his angular parameter \(\phi\) may be set equal to zero with no loss of generality) that if \(\alpha\) lies on the unit circle \(U\) (i.e. \(|\alpha| = 1\)) and the zeros of \(g(z)\) lie in a vertical strip \(S\), then the zeros of \(T(\alpha)g(z)\) lie in the same strip \(S\).

Now define an operator \(\Delta_i\) by

\[
\Delta_i g(z) = g(z + \lambda_i) - g(z).
\]

**Lemma.** If all zeros of the nonconstant polynomial \(g(z)\) lie on \(\text{Re} z = \sigma\), then all zeros of \(\Delta_i g(z)\) lie on

\[
\text{Re} z = \sigma - (\lambda_i/2).
\]

**Proof.** Let \(s = z + (\lambda_i/2)\). Then

\[
\Delta_i g(z) = g(s + (\lambda_i/2)) - g(s - (\lambda_i/2)).
\]

Since all zeros of \(g(s)\) have real part \(\sigma\), the result follows from the case \(\alpha = -1\) of Obrechkoff’s theorem.

**Corollary.** All zeros of

\[
\Delta_1 \Delta_2 \cdots \Delta_J z^n = 0
\]

lie on

\[
\text{Re} z = -\left(\sum_{j=1}^{J} \lambda_j\right)/2.
\]

**Proof.** Apply the above lemma \(J\) times with \(\sigma = 0\). The above lemma can also be deduced from a lemma in [T, p. 238].

3. Proof of the theorem. Clearly

\[
R_n(z) := P_n E(z) = 1 - \sum_{j} \left(1 + \frac{\lambda_j z}{n}\right)^n + \sum_{i \neq j} \left(1 + \frac{(\lambda_i + \lambda_j) z}{n}\right)^n - \ldots.
\]
Set \( w = n/z \). Thus
\[
(3.2) \quad w^n R_n(n/w) = w^n - \sum_j (w + \lambda_j)^n + \sum_{i<j} (w + \lambda_i + \lambda_j)^n - \cdots
\]
where the signs alternate and (consider \( w^n \) as the 0th sum on the right) the \( k \)th sum has the form
\[
(3.3) \quad \sum (w + \lambda(j_1) + \cdots + \lambda(j_k))^n
\]
where \( \lambda(j) = \lambda_j \) and the sum is over all \( k \)-tuples
\[
(3.4) \quad j_1 < \cdots < j_k.
\]
It is now clear that
\[
(3.5) \quad w^n R_n(n/w) = \Delta_1 \Delta_2 \cdots \Delta_J w^n,
\]
and that the right-hand side is not identically zero unless \( J > n \). By the previous corollary,
\[
(3.6) \quad R_n(z) = 0
\]
implies
\[
(3.7) \quad \text{Re } w = -\left( \sum \lambda_j \right)/2
\]
so \( z = n/w \) lies on a circle through the origin that is symmetric with respect to the real axis, and cuts the real axis at
\[
(3.8) \quad x_0 = -2n/\sum \lambda_j.
\]
Hence \( z \) must lie on a circle of radius \( |r| \) and center \(-r\) where
\[
(3.9) \quad r = n/\sqrt{\sum_{j=1}^J \lambda_j}.
\]
If \( r \) is infinite, the above argument shows that the zeros all lie on the imaginary axis. This completes the proof.

**Remark.** Define the operators \( \Delta \) and \( B_J \) by
\[
(3.10) \quad \Delta F(n) = F(n+1) - F(n)
\]
and
\[
(3.11) \quad B_J F(n) = J!^{-1} \Delta^J F(n)
\]
for any number theoretic function \( F = F(n) \). Thus,
\[
(3.12) \quad B_J F(n) = \frac{1}{J!} \sum_{k=0}^J (-1)^{J-k} \binom{J}{k} F(n+k).
\]
If \( F = F_u(n) \) is the function \( n^u \), where \( u \) is a nonnegative integer, then it is well known that \( B_J F_u(0) \) is a Stirling number of the second kind, and
\[
(3.13) \quad B_J F_u(0) = \delta(J, u), \quad 0 \leq u \leq J,
\]
where $\delta(i,j)$ is the Kronecker delta. Professor Graydon Bell of the University of Northern Arizona has pointed out to us that

$$B_j P_n e^z = \sum_{k=0}^J \frac{z^k}{k!}.$$  

Hence the operator $B_j$ provides a link between the two most common approximations to $e^z$. Formula (3.14) can be deduced from (3.13). Also, it can be inverted to yield

$$\left(1 + \frac{z}{n}\right)^n = \sum_{k=0}^n k n^{-k-1} \frac{n!}{(n-k)!} \sum_{j=0}^k \frac{z^j}{j!}.$$  

References


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