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A NOTE ON THE "HYPERBOLIC" BOCHNER-RIESZ MEANS

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ABSTRACT. We consider the $L^p(\mathbf{R}^2)$ boundedness properties of the Fourier multiplier $m(\xi_1, \xi_2) = (1 - \xi_1^2\xi_2^2)_+^\alpha$ for $\alpha > 0$. We prove that if $\alpha \geq \frac{1}{2}$, then m is bounded on L^p, $1 < p < \infty$, and that if $\alpha > 0$, then m is bounded on L^p, $\frac{4}{3} \leq p \leq 4$.

1. Introduction. In [**6**], El-Kohen defined the hyperbolic Bochner-Riesz multiplier of order λ for $\mathbf{R}^2$ as

$$m^\lambda(\xi_1, \xi_2) = (1 - (\xi_1\xi_2)^2)_+^\lambda.$$

If $(T^\lambda f)\widehat{\ }(\xi) = m^\lambda(\xi)\hat{f}(\xi)$, he showed that $\|T^\lambda f\|_p \leq C_{p,\lambda}\|f\|_p$ for $1 < p < \infty$ provided that $\lambda > 1$ (a result which may be deduced from the Marcinkiewicz multiplier theorem; see Stein [**8**]). He also conjectured that the critical index $\lambda_0 = \inf\{\lambda > 0 | \|T^\lambda f\|_p \leq C_{p,\lambda}\|f\|_p,\ 1 < p < \infty\}$ was indeed one. In this note we prove that, on the contrary, the hyperbolic Bochner-Riesz means behave like the usual spherical ones, so that T^λ is bounded on L^p, $1 < p < \infty$, provided that $\lambda > \frac{1}{2}$, and that T^λ is bounded on L^p, $\frac{4}{3} \leq p \leq 4$, provided that $\lambda > 0$.

THEOREM 1. *If $1 < p < \infty$ and $\lambda > 0$, then*

$$\|T^\lambda f\|_p \leq C_{p,\lambda}\|f\|_p$$

provided that $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2}(\lambda + \frac{1}{2})$.

In proving Theorem 1 we may assume that $\operatorname{supp}\hat{f}$ is contained in the first quadrant, and to facilitate calculations we shall sometimes employ the rectangular hyperbolic coordinates $\alpha = \xi_1\xi_2$ and $\beta = \xi_1^2 - \xi_2^2$ in Fourier transform space, so that $(T^\lambda f)\widehat{\ }(\alpha, \beta) = (1 - \alpha^2)_+^\lambda \hat{f}(\alpha, \beta)$. We make the usual decomposition $(1 - \alpha^2)_+^\lambda = \sum_{k=0}^\infty 2^{-k\lambda}\phi_k(\alpha)$, with ϕ_k a smooth nonnegative bump function supported in $[1 - 2^{-k}, 1 - 2^{-k-2}]$. If $(T_k f)\widehat{\ }(\alpha, \beta) = \phi_k(\alpha)\hat{f}(\alpha, \beta)$, we see that it suffices to obtain estimates for the form

$$\|T_k f\|_p \leq C_p(1 + k)^\rho 2^{k(1-4/p)/2}\|f\|_p$$

(for some constant $\rho > 0$) when $4 \leq p < \infty$. Now ϕ_0 is easily seen to satisfy the hypotheses of the Marcinkiewicz multiplier theorem [**8**] and we turn to the estimates for T_k, $k \geq 1$.

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2. Proof of Theorem 1. As outlined above, Theorem 1 will follow from

THEOREM 2. *Let Φ be a fixed C^∞ function supported in $[-1,1]$. For $0<\delta<1$ and $\alpha>0$, let $\phi(\alpha)=\Phi((\alpha-1)/\delta)$. Let $(Tf)\hat{}(\alpha,\beta)=\phi(\alpha)\hat{f}(\alpha,\beta)$. Then*

$$\|Tf\|_p \le C_p\delta^{-1/2}\|f\|_p, \qquad 1<p<\infty,$$

and

$$\|Tf\|_4 \le C\left(\log\frac{1}{\delta}\right)^\rho \|f\|_4,$$

where C_p and C depend only on $\max_{0\le j\le 3}\|\Phi^{(j)}\|_\infty$.

The proof of Theorem 2 follows the argument given by Córdoba in [**3**] to prove the Carleson and Sjölin theorem [**1**] concerning the spherical Bochner-Riesz multipliers. We shall assume that $\delta^{-1/2}\in\mathbf{N}$; for $-\infty<k<\infty$ and $0\le j<\delta^{-1/2}-1$ let $S_j^k=\{(\xi_1,\xi_2)\in\mathbf{R}^2|2^k(1+\delta^{-1/2}j)\le\xi_1\le 2^k(1+\delta^{-1/2}(j+1))\}$, let $\{\gamma_j^k\}$ be a smooth partition of unity subordinate to $\{S_j^k\}$, and let $\tilde\gamma_j^k$ be a smooth function such that $\tilde\gamma_j^k\equiv 1$ on S_j^k, and vanishes outside the two immediate neighbours of S_j^k. (We choose γ_j^k and $\tilde\gamma_j^k$ to depend only on ξ_1.) Let $(A_j^kf)\hat{}=\gamma_j^k\hat f$ and $(B_j^kf)\hat{}=\tilde\gamma_j^k\hat f$. Theorem 2 will follow from the following three propositions and a simple interpolation argument.

PROPOSITION 1. (a) $\|Tf\|_p\le C_p\delta^{-1/4}\|(\sum_{k,j}|A_j^kTf|^2)^{1/2}\|_p$, $2\le p<\infty$.
(b) $\|Tf\|_4\le C(\log\frac{1}{\delta})^\rho\|(\sum_{k,j}|A_j^kTf|^2)^{1/2}\|_4$.

PROPOSITION 2. (a) $\|(\sum_{k,j}|TA_j^kf|^2)^{1/2}\|_p\le C_\Phi C_p\delta^{-1/4}\|(\sum_{k,j}|B_j^kf|^2)^{1/2}\|_p$, $2\le p<\infty$.
(b) $\|(\sum_{k,j}|TA_j^kf|^2)^{1/2}\|_4\le C_\Phi(\log\frac{1}{\delta})^\rho\|(\sum_{k,j}|B_j^kf|^2)^{1/2}\|_4$.

PROPOSITION 3. $\|(\sum_{k,j}|B_j^kf|^2)^{1/2}\|_p\le C_p\|f\|_p$, $2\le p<\infty$.

Proposition 3 follows from Córdoba's scissors lemma [**4**], and Propositions 1 and 2 follow from simple geometric observations about $\{S_k^j\cap\operatorname{supp}\phi\}$ together with an application of maximal theorems due to Nagel, Stein and Wainger and to Córdoba. (We shall assume without loss of generality that $k\ge 0$ when proving these propositions.)

PROOF OF PROPOSITION 1. (a) By the theory of vector-valued singular integrals [**8**]

$$\|Tf\|_p\le C_p\left\|\left(\sum_k\left|\sum_{j=0}^{\delta^{-1/2}-1}A_j^kTf\right|^2\right)^{1/2}\right\|_p\le C_p\delta^{-1/4}\left\|\left(\sum_{k,j}\left|A_j^kTf\right|^2\right)^{1/2}\right\|_p,$$

(b)

$$\int |Tf|^4 = \int |\widehat{Tf} * \widehat{Tf}|^2 = \int \left| \sum_{k,j} \sum_{k',j'} \gamma_j^k \phi \hat{f} * \gamma_{j'}^{k'} \phi \hat{f} \right|^2$$

$$\le C \int \sum_{k,j} \sum_{k',j'} |\gamma_j^k \phi \hat{f} * \gamma_{j'}^{k'} \phi \hat{f}|^2 \tag{1}$$

$$= C \int \sum_{k,j} \sum_{k',j'} |A_j^k Tf \cdot A_{j'}^{k'} Tf|^2 = C \int \left(\sum_{k,j} |A_j^k Tf|^2 \right)^2 . \quad \square$$

Inequality (1) holds because of the following

LEMMA 1. *No point of* $\mathbf{R}^2$ *belongs to more than* C *of the sets* $\{(S_k^j \cap \operatorname{supp} \phi) + (S_{k'}^{j'} \cap \operatorname{supp} \phi)\}$.

PROOF. The lemma is true because of two properties: the global property that no point in $\mathbf{R}^2$ can be written in more than two ways as a sum of points in the first quadrant and lying on the rectangular hyperbola $\xi_1 \xi_2 = 1$ and the local property that if $y = (\xi_1^j, \xi_2^j) + (\mu_1^j, \mu_2^j)$ with $|\xi_1^j \xi_2^j - 1| < \delta$ and $|\mu_1^j \mu_2^j - 1| < \delta$, $\xi_1^j \le \mu_1^j$, $j = (1,2)$, then $|1 - \xi_1^1/\xi_1^2| \le c\delta^{1/2}$. To see this latter fact observe that if we perturb the solutions of the equation $y = (\xi_1, \xi_2) + (\mu_1, \mu_2)$, $\xi_1 \xi_2 = 1 = \mu_1 \mu_2$ by allowing $|\xi_1 \xi_2 - 1|$ and $|\mu_1 \mu_2 - 1| < \delta$, then ξ_1 and μ_1 will change by a factor no larger than $\delta(\xi_1 + \mu_1)/|\xi_1 \mu_2 - \xi_2 \mu_1|$. Thus the "marginal" change in ξ_1 and μ_1 is less than $\delta^{1/2}$ provided that $|1 + \lambda|/|\lambda - 1/\lambda| \le \delta^{-1/2}$, where $\lambda = \xi_1/\mu_1$ or μ_1/ξ_1. If $|1 + \lambda|/|\lambda - 1/\lambda| \ge \delta^{-1/2}$, however, then automatically $|1 - \lambda| \le c\delta^{1/2}$. $\square$

Let M be the maximal function corresponding to averages over rectangles having a side parallel of one of the chords P_{kj} which joins the two points of $\partial S_j^k \cap \{\xi_1 \xi_2 = 1\}$. Concerning M we have the following known results.

LEMMA 2. (a) (*Córdoba* [**2**]) $\|M\omega\|_2 \le C(\log \frac{1}{\delta})^\rho \|\omega\|_2$.
(b) (*Nagel, Stein and Wainger* [**7**]) $\|M\omega\|_q \le C_q \delta^{-1/2} \|\omega\|_q$, $1 < q < \infty$. $\square$

To prove Proposition 2 we also need

LEMMA 3. *The kernel* L_j^k *of the multiplier operator* TA_j^k *satisfies*
(i) $\int |L_j^k| \le C$,
(ii) $|L_j^k| * g(x) \le CMg(x)$.
Here, C *depends only on* $\max_{0 \le j \le 3} \|\Phi^{(j)}\|_\infty$.

PROOF. We need only see that $\phi\gamma_j^k$ is a smooth bump function supported in a rectangle with width $10\delta/2^k$ and length $10\delta^{1/2}2^k$ with longer side parallel to P_{kj}. This is clear since the distance between P_{kj} and the hyperbola $\xi_1\xi_2 = 1$ never exceeds $3\delta/2^k$. We now apply the integration by parts procedure as in [**3**] to obtain our result. $\square$

PROOF OF PROPOSITION 2. For $p \geq 2$, let $\|\omega\|_{(p/2)'} \leq 1$. Then, following [**3**],

$$\int \sum_{k,j} |TA_j^k f|^2 \omega = \int \sum_{k,j} |L_j^k * B_j^k f|^2 \omega \leq C \int \sum_{k,j} |L_j^k| * |B_j^k f|^2(x)\omega(x)\, dx$$
$$= C \int \sum_{k,j} |B_j^k f|^2(x) |L_j^k| * \omega(x)\, dx \leq C \int \sum_{k,j} |B_j^k f|^2 M\omega.$$

Lemma 2 now completes the proof of this proposition. □

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