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# DOUBLE COMMUTANTS OF OPERATORS QUASI-SIMILAR TO NORMAL OPERATORS

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ABSTRACT. It is shown that double commutants of operators quasi-similar to normal operators are reflexive.

An algebra $\mathscr{A}$ of bounded linear operators on a Hilbert space is said to be *reflexive* if $\mathscr{A} = \text{Alg Lat } \mathscr{A}$, where Lat $\mathscr{A}$ denotes the family of all subspaces invariant under all elements of $\mathscr{A}$ and Alg Lat $\mathscr{A}$ is the algebra of all operators $X$ for which $X\mathscr{M} \subseteq \mathscr{M}$ for every $\mathscr{M} \in \text{Lat } \mathscr{A}$. If $\mathscr{A}$ is a von Neumann algebra, then $\mathscr{M} \in \text{Lat } \mathscr{A}$ means that the projection to $\mathscr{M}$ belongs to the commutant $\mathscr{A}'$, therefore Alg Lat $\mathscr{A}$ coincides with the double commutant $\mathscr{A}''$. Thus the von Neumann double commutant theorem can be rephrased as follows (cf. [**4**, Theorem 9.17]): every von Neumann algebra is reflexive.

Recall some definitions. For operators $T_1$ and $T_2$, let us write $T_1 \prec T_2$ (resp. $T_1 \overset{i}{\prec} T_2$) to denote that there exists a *quasi-affinity*, i.e. injection with dense range (resp. an injection) $X$ such that $XT_1 = T_2X$. $T_1$ and $T_2$ are *quasi-similar* by definition if $T_1 \prec T_2$ and $T_2 \prec T_1$.

If $N$ is a normal operator, the commutant $\{N\}'$ becomes a von Neumann algebra by the Fuglede theorem, so that both the commutant and the double commutant $\{N\}''$ are reflexive. If an operator $T$ is quasi-similar to a normal operator $N$, the commutant $\{T\}'$ is reflexive because the reflexivity of commutant is preserved under quasi-similarity [**1**] and $\{N\}'$ is reflexive. In this note we shall show that the double commutant $\{T\}''$ is reflexive too.

THEOREM. *If an operator $T$ is quasi-similar to a normal operator $N$, then the double commutant $\{T\}''$ is reflexive.*

Before going into the proof, let us present an immediate consequence.

COROLLARY. *If $T$ is a contraction of class $C_{11}$, that is, $\lim_{n\to\infty}\|T^n x\| \neq 0$ and $\lim_{n\to\infty}\|T^{*n}x\| \neq 0$ for every nonzero $x$, then the double commutant $\{T\}''$ is reflexive.*

In fact, it is well known that a contraction of class $C_{11}$ is quasi-similar to a unitary operator (see [**5**, Proposition II.3.5]).

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This result for a $C_{11}$-contraction $T$ was proved in [**3**, Theorem 3] under the condition that every injection in $\{T\}'$ has dense range.

For the proof of the theorem we need a lemma, which is an analog of [**3**, Lemma 5] for $C_{11}$-contractions.

LEMMA. *Let $T$ be an operator quasi-similar to a normal operator. Then, for $\mathscr{M} \in \operatorname{Lat} T$ (= the family of all $T$-invariant subspaces), the following conditions are mutually equivalent.*

(i) *$T|\mathscr{M}$ is quasi-similar to a normal operator.*

(ii) *$T|\mathscr{M} \succ N_1$ for some normal operator $N_1$.*

(iii) *$\mathscr{M} = (\operatorname{ran} Z)^-$ for some $Z \in \{T\}'$.*

*If one of these conditions holds, then $\mathscr{M} \in \operatorname{Lat}\{T\}''$.*

PROOF. Let $X$ and $Y$ be quasi-affinities such that $XT = NX$ and $TY = YN$ for some normal operator $N$ on $\mathscr{G}$. (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii): Since $N_1 \prec T|\mathscr{M} \prec N|(X\mathscr{M})^-$, it follows from [**2**, Lemma 4.1] that $(X\mathscr{M})^-$ is $N$-reducing and $N|(X\mathscr{M})^-$ is unitarily equivalent to $N_1$. Therefore we have $(T|\mathscr{M})^* \prec N_1^* \prec N^*|(X\mathscr{M})^- \underset{i}{\prec} N^* \prec T^*$. Denote by $W$ the injection from $\mathscr{M}$ to $\mathscr{H}$ (= the space on which $T$ acts) such that $W(T|\mathscr{M})^* = T^*W$, and by $J$ the inclusion mapping of $\mathscr{M}$ into $\mathscr{H}$. Then $Z = JW^*$ is an operator required in (iii).

(iii) $\Rightarrow$ (i): Define an operator $K$ from $\mathscr{G}$ to $\mathscr{M}$ by $Kx = ZYx$ for $x \in \mathscr{G}$. Then $K$ has dense range and $(T|\mathscr{M})K = KN$. Also $(X|\mathscr{M})(T|\mathscr{M}) = N(X|\mathscr{M})$ and $X|\mathscr{M}$ is injective. Therefore it follows from [**2**, Lemma 4.1] that $N|(\ker K)^\perp$ and $N|(X\mathscr{M})^-$ are unitarily equivalent normal operators, and so $T|\mathscr{M}$ is quasi-similar to a normal operator $N|(X\mathscr{M})^-$. Finally it is clear that $\mathscr{M} \in \operatorname{Lat}\{T\}''$ for any subspace $\mathscr{M}$ satisfying (iii).

PROOF OF THEOREM. By assumption there are quasi-affinities $X$ and $Y$ satisfying $XT = NX$ and $TY = YN$. Without loss of generality, the product $XY$ can be assumed to be nonnegative. Indeed, since $XY \in \{N\}'$, $XY$ admits the polar factorization $XY = UP$ in the von Neumann algebra $\{N\}'$, where $U$ is unitary and $P$ is nonnegative. The quasi-affinities $U^*X$ and $Y$ satisfy the required condition.

The inclusion $\{T\}'' \subseteq \operatorname{Alg} \operatorname{Lat}\{T\}''$ is obvious. To see the converse inclusion, take $A \in \operatorname{Alg} \operatorname{Lat}\{T\}''$. In order to show $A \in \{T\}''$, it suffices to prove that $A$ commutes with $YCX$ for all $C \in \{N\}'$. In fact, then $A$ commutes with $YX$, and also with $YXBYX$ for any $B \in \{T\}'$, because $XBY$ belongs to $\{N\}'$. Therefore we have

$$YXBAYX = YXBYXA = AYXBYX = YXABYX.$$

Then the quasi-affinity properties of $X$ and $Y$ imply $BA = AB$, hence $A \in \{T\}''$.

Let us show that $A$ commutes with $YCX$ for all $C \in \{N\}'$. Since $\{N\}'$ is a von Neumann algebra, we have only to show that $A(YHX) = (YHX)A$ for any selfadjoint injection $H \in \{N\}'$. Take $\mathscr{M} \in \operatorname{Lat}\{XYHXY\}'$. Since $\{N\}'' \subseteq \{XYHXY\}'$, we have $\mathscr{M} \in \operatorname{Lat}\{N\}''$, hence $\mathscr{M}$ is $N$-reducing and $N|\mathscr{M}$ is normal. Then, since $(Y\mathscr{M})^- \in \operatorname{Lat} T$ and $T|(Y\mathscr{M})^- \succ N|\mathscr{M}$, it follows from the Lemma that $(Y\mathscr{M})^- \in \operatorname{Lat}\{T\}'' \subseteq \operatorname{Lat} A$, and so $XYHXAY\mathscr{M} \subseteq (XYHXY\mathscr{M})^- \subseteq \mathscr{M}$. We can conclude that $XYHXAY \in \operatorname{Alg} \operatorname{Lat}\{XYHXY\}'$. Then, since the commutant $\{XYHXY\}'$ of the

selfadjoint operator $XYHXY$ is reflexive, $XYHXAY \in \{XYHXY\}'$. Taking $H = I$, we have $(XY)^2XYXAY = XYXAY(XY)^2$ and $(XY)^2XAY = XAY(XY)^2$ by the injectivity of $XY$. Since $XY$ is assumed to be nonnegative, taking the square root of $(XY)^2$, we obtain $XYXAY = XAYXY$. Then for any selfadjoint injection $H$ we have

$$XYHXYXYHXAY = XYHXAYXYHXY = XYHXYXAYHXY.$$

Finally it follows from the injectivity of $XYHXYX$ and the dense range property of $Y$ that $YHXA = AYHX$.

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