PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 92, Number 3, November 1984

A SIMPLE EVALUATION OF ASKEY AND WILSON'S q-BETA INTEGRAL

MIZAN RAHMAN[1]

ABSTRACT. By using the well-known sum of ${}_2\phi_1(a, b; c; c/ab)$ and Sears' identity for the sum of two nonterminating balanced ${}_3\phi_2$ series, a simple evaluation is given for Askey and Wilson's q-beta type integral

$$\int_{-1}^{1} \frac{h(x;1)h(x;-1)h(x;\sqrt{q})h(x;-\sqrt{q})}{h(x;a)h(x;b)h(x;c)h(x;d)} \frac{dx}{\sqrt{1-x^2}},$$

where $\max(|q|, |a|, |b|, |c|, |d|) < 1$.

1. Introduction. The classical beta-integral

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \tag{1.1}$$

has been extended in a number of different ways. Thomae's **[8]** original extension, namely,

$$\frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)} = B_q(x, y) = \int_0^1 t^{x-1}(qt)_{y-1}\,d_q t, \tag{1.2}$$

used the q-integral

$$\int_0^a f(x)\,d_q x = a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n, \tag{1.3}$$

$$\int_a^b f(x)\,d_q x = \int_0^b f(x)\,d_q x - \int_0^a f(x)\,d_q x.$$

This is another way of writing the q-binomial theorem

$$\frac{(ay)_\infty}{(y)_\infty} = \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} y^n. \tag{1.4}$$

The q-gamma function can be defined by **[3]**

$$\Gamma_q(x) = \frac{(q)_\infty}{(q^x)_\infty}(1-q)^{1-x}. \tag{1.5}$$

Received by the editors November 7, 1983. The contents of this paper were presented at the 810th Meeting of the American Mathematical Society at the University of Notre Dame, Notre Dame, Indiana, on April 6, 1984.

1980 *Mathematics Subject Classification*. Primary 33A15, 33A65.

Key words and phrases. q-beta integral, q-binomial theorems, Sears' identity, Askey and Wilson's integral.

[1]This research was supported by NSERC Grant A6197.

©1984 American Mathematical Society
0002-9939/84 $1.00 + $.25 per page

The notation $(a)_n$ is an abbreviation for $(a;\, q)_n$ which is defined by

$$(a;\, q)_n = \frac{(a;q)_\infty}{(aq^n:q)_\infty}, \tag{1.6}$$

whether or not n is an integer, where

$$(a)_\infty = (a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n), \tag{1.7}$$

$0 < q < 1$, and no factor in $(aq^n;\, q)_\infty$ vanishes.

Andrews and Askey [**2**] needed a more general extension of (1.1) than (1.2) to serve as the weight function of a set of orthogonal polynomials, so were led to

$$\int_a^b \frac{(qt/a)_\infty (qt/b)_\infty}{(dt)_\infty (et)_\infty} d_q t = \frac{b(1-q)(q)_\infty (bq/a)_\infty (a/b)_\infty (abde)_\infty}{(ad)_\infty (ae)_\infty (bd)_\infty (be)_\infty} \tag{1.8}$$

subject to the restriction that there are no zero factors in the denominators.

Al-Salam and Verma [**1**] pointed out that (1.8) is a special case of a more general q-integral

$$\begin{aligned} &\int_a^b \frac{(qt/a)_\infty (qt/b)_\infty (ct)_\infty}{(dt)_\infty (et)_\infty (ft)_\infty} d_q t \\ &\quad = \frac{b(1-q)(q)_\infty (bq/a)_\infty (a/b)_\infty (c/d)_\infty (c/e)_\infty (c/f)_\infty}{(ad)_\infty (ae)_\infty (af)_\infty (bd)_\infty (be)_\infty (bf)_\infty}, \end{aligned} \tag{1.9}$$

where $c = abdef$, which is simply another way of writing Sears' identity [**6**, (5.2)] for the sum of two nonterminating balanced ${}_3\phi_2$ series.

Very recently another q-extension was given by Askey and Wilson [**5**]

$$\begin{aligned} I(a,b,c,d) &\equiv \frac{1}{2\pi} \int_{-1}^{1} \frac{h(x;1)h(x;-1)h(x;\sqrt{q})h(x;-\sqrt{q})}{h(x;a)h(x;b)h(x;c)h(x;d)} \frac{dx}{\sqrt{1-x^2}} \\ &= \frac{(abcd)_\infty}{(q)_\infty (ab)_\infty (ac)_\infty (ad)_\infty (bc)_\infty (bd)_\infty (cd)_\infty} \end{aligned} \tag{1.10}$$

provided $|q| < 1$ and $\max(|a|, |b|, |c|, |d|) < 1$, where

$$\begin{aligned} h(x;\, a) &= \prod_{n=0}^{\infty} (1 - 2axq^n + a^2 q^{2n}) \\ &= (ae^{i\theta})_\infty (ae^{-i\theta})_\infty \quad \text{if } x = \cos\theta. \end{aligned} \tag{1.11}$$

Unlike (1.2) and (1.8), $I(a, b, c, d)$ is not a q-integral, but a Riemann integral. By setting $a = -b = \sqrt{q}$, $c = q^{\alpha+1/2}$, $d = -q^{\beta+1/2}$ in (1.10) and using (1.5) one can show that (1.10) reduces to (1.1) in the limit $q \to 1$. However, Askey and Wilson's original proof of (1.10) is far from elementary. They used a contour integration and had to make a number of assumptions that had to be removed later. They gave a simpler evaluation of the reduced integral

$$(1.12)\quad J(a,b)=\frac{1}{2\pi}\int_{-1}^{1}\frac{h(x;1)h(x;-1)}{h(x;a)h(x;b)}\frac{dt}{\sqrt{1-x^2}}$$
$$=\frac{(-abq)_\infty}{(q)_\infty(-q)_\infty(a\sqrt{q})_\infty(-a\sqrt{q})_\infty(b\sqrt{q})_\infty(-b\sqrt{q})_\infty(ab)_\infty}$$

by using the known sums of the bilateral series ${}_1\psi_1$ and ${}_4\psi_4$. Unhappy that this procedure does not seem to work for the full four-parameter integral $I(a,b,c,d)$, Askey [**4**] published another proof of (1.10) by a very elementary method. Unfortunately, as Askey himself observed in [**4**], the proof is not really an evaluation of $I(a,b,c,d)$, rather a verification of (1.10). Since (1.10) is a very attractive result and contains a large number of known and previously unknown formulas as special cases (see [**5**] for a full discussion), a simpler proof based on a minimum of restrictions seems like a reasonable goal.

In this note we offer a proof that makes repeated use of (1.9) and uses very few assumptions.

2. Proof of (1.10). The first step is to recognize that the integrand in (1.10) is an even function of θ, where $x=\cos\theta$, so that we can write

$$(2.1)\qquad I(a,b,c,d)=\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{h(x;1)h(x;-1)h(x;\sqrt{q})h(x;-\sqrt{q})}{h(x;a)h(x;b)h(x;c)h(x;d)}\,d\theta.$$

Let us now replace c, d and e in (1.9) by 1, $e^{i\theta}$ and $e^{-i\theta}$, respectively, to get

$$(2.2)\qquad \frac{h(x;1)}{h(x;a)h(x;b)}=\frac{(a^{-1})_\infty(b^{-1})_\infty}{b(1-q)(q)_\infty(bq/a)_\infty(a/b)_\infty(ab)_\infty}\cdot\int_a^b d_qu\frac{(qu/a)_\infty(uq/b)_\infty(u)_\infty}{(u/ab)_\infty}\frac{1}{h(x;u)}$$

provided $a\neq b$ and a nor b is of the form q^j, $j=1,2,\ldots$. This is not an essential restriction because if either of a, b is of this form then we may choose $c=\sqrt{q}$ or $-\sqrt{q}$ that will produce $h(x;\pm\sqrt{q})$ on the left instead of $h(x;1)$ and $(\sqrt{q}\,a^{-1})_\infty(\sqrt{q}\,b^{-1})_\infty$ on the right. Next, we replace c, d and e in (1.9) by -1, $e^{i\theta}$ and $e^{-i\theta}$, and let a, b be replaced by c, d, respectively. This gives

$$(2.3)\qquad \frac{h(x;-1)}{h(x;c)h(x;d)}=\frac{(-c^{-1})_\infty(-d^{-1})_\infty}{d(1-q)(q)_\infty(dq/c)_\infty(c/d)_\infty(cd)_\infty}\cdot\int_c^d d_qv\frac{(qv/c)_\infty(qv/d)_\infty(-v)_\infty}{(-v/cd)_\infty}\frac{1}{h(x;v)}.$$

Finally we replace a, b, c, d and e in (1.9) by $u/\sqrt{q}$, $v/\sqrt{q}$, $-q$, $\sqrt{q}\,e^{i\theta}$ and $\sqrt{q}\,e^{-i\theta}$, respectively, and get

$$(2.4)\qquad \frac{h(x;-\sqrt{q})}{h(x;u)h(x;v)}=\frac{\sqrt{q}(-\sqrt{q}/u)_\infty(-\sqrt{q}/v)_\infty}{v(1-q)(q)_\infty(vq/u)_\infty(u/v)_\infty(uv)_\infty}\cdot\int_{u/\sqrt{q}}^{v/\sqrt{q}}d_qt\frac{(tq^{3/2}/u)_\infty(tq^{3/2}/v)_\infty(-qt)_\infty}{(-tq/uv)_\infty}\cdot\frac{1}{h(x;\sqrt{q}\,t)}.$$

Now,

$$(2.5)\qquad \frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{h(x;\sqrt{q})}{h(x;t\sqrt{q})}d\theta=\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{(\sqrt{q}e^{i\theta})_\infty(\sqrt{q}e^{-i\theta})_\infty}{(t\sqrt{q}e^{i\theta})_\infty(t\sqrt{q}e^{-i\theta})_\infty}d\theta$$

$$=\frac{1}{4\pi}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(t^{-1})_k(t^{-1})_l}{(q)_k(q)_l}(\sqrt{q}t)^{k+l}$$

$$\cdot\int_{-\pi}^{\pi}e^{i(k-l)\theta}\,d\theta\quad\text{by (1.4)}$$

$$=\frac{1}{2}\sum_{k=0}^{\infty}\frac{(t^{-1})_k(t^{-1})_k}{(q)_k(q)_k}(qt^2)^k$$

$$=\frac{1}{2}{}_2\phi_1\left[\begin{matrix}t^{-1},t^{-1}\\ q\end{matrix};qt^2\right]$$

$$=\frac{(qt)_\infty(qt)_\infty}{2(q)_\infty(qt^2)_\infty}$$

by [7, (IV.2), p. 247], if $|\sqrt{q}\,t|<1$. This inequality is clearly satisfied since by (2.4), $\sqrt{q}\,t$ equals either u or v which, in turn, attains values among a, b, c and d all of whom are assumed to be numerically less than 1.

However,

(2.6)

$$(qt^2;q)_\infty=(qt^2;q^2)_\infty(q^2t^2;q^2)_\infty=(\sqrt{q}t;q)_\infty(-\sqrt{q}t;q)_\infty(qt;q)_\infty(-qt;q)_\infty.$$

Hence the r.h.s. of (2.5) equals $(qt)_\infty/2(q)_\infty(-qt)_\infty(\sqrt{q}\,t)_\infty(-\sqrt{q}\,t)_\infty$. Combining (2.2)–(2.5) we obtain

(2.7)

$$I(a,b,c,d)=\frac{(a^{-1})_\infty(b^{-1})_\infty(-c^{-1})_\infty(-d^{-1})\sqrt{q}}{2bd(1-q)^3(q)_\infty^4(bq/a)_\infty(a/b)(ab)_\infty(dq/c)_\infty(c/d)_\infty(cd)_\infty}$$

$$\cdot\int_a^b d_qu\frac{(qu/a)_\infty(qu/b)_\infty(u)_\infty}{(u/ab)_\infty}$$

$$\cdot\int_c^d d_qv\frac{(qv/c)_\infty(qv/d)_\infty(-v)_\infty}{(-v/cd)_\infty}\frac{(-\sqrt{q}/u)_\infty(-\sqrt{q}/v)_\infty}{v(vq/u)_\infty(u/v)_\infty(uv)_\infty}$$

$$\cdot\int_{u/\sqrt{q}}^{v/\sqrt{q}}d_qt\frac{(tq^{3/2}/u)_\infty(tq^{3/2}/v)_\infty(qt)_\infty}{(\sqrt{q}t)_\infty(-\sqrt{q}t)_\infty(-qt/uv)_\infty}.$$

By (1.9) the last q-integral over t equals

$$\frac{v/\sqrt{q}(1-q)(q)_\infty(vq/u)_\infty(u/v)_\infty(\sqrt{q})_\infty(-\sqrt{q})_\infty(-uv)_\infty}{(u)_\infty(v)_\infty(-u)_\infty(-v)_\infty(-\sqrt{q}/u)_\infty(-\sqrt{q}/v)_\infty}.$$

Hence

(2.8)

$$I(a,b,c,d)=\frac{(a^{-1})_\infty(b^{-1})_\infty(-c^{-1})_\infty(-d^{-1})_\infty(\sqrt{q})_\infty(-\sqrt{q})_\infty}{2bd(1-q)^2(q)_\infty^3(bq/a)_\infty(a/b)_\infty(ab)_\infty(dq/c)_\infty(c/d)_\infty(cd)_\infty}$$
$$\cdot\int_a^b d_qu\frac{(qu/a)_\infty(qu/b)_\infty}{(-u)_\infty(u/ab)_\infty}\cdot\int_c^d d_qv\frac{(qv/c)_\infty(qv/d)_\infty(-uv)_\infty}{(v)_\infty(uv)_\infty(-v/cd)_\infty}.$$

The q-integral over v equals

$$\frac{d(1-q)(q)_\infty(dq/c)_\infty(c/d)_\infty(-1)_\infty(-u)_\infty(cdu)_\infty}{(c)_\infty(d)_\infty(-c^{-1})_\infty(-d^{-1})_\infty(cu)_\infty(du)_\infty}.$$

Thus

$$(2.9)\quad I(a,b,c,d)=\frac{(a^{-1})_\infty(b^{-1})_\infty(\sqrt{q})_\infty(-\sqrt{q})_\infty(-1)_\infty}{2b(1-q)(q)_\infty^2(bq/a)_\infty(a/b)_\infty(ab)_\infty(cd)_\infty(c)_\infty(d)_\infty}$$
$$\cdot\int_a^b d_qu\frac{(qu/a)_\infty(qu/b)_\infty(cdu)_\infty}{(cu)_\infty(du)_\infty(u/ab)_\infty}.$$

One final application of (1.9) gives the value of the integral above as

$$\frac{b(1-q)(q)_\infty(bq/a)_\infty(a/b)_\infty(d)_\infty(c)_\infty(abcd)_\infty}{(ac)_\infty(ad)_\infty(b^{-1})_\infty(bc)_\infty(bd)_\infty(a^{-1})_\infty}.$$

Hence

$$(2.10)\qquad I(a,b,c,d)=\frac{(-1)_\infty(\sqrt{q})_\infty(-\sqrt{q})_\infty(abcd)_\infty}{2(q)_\infty(ab)_\infty(ac)_\infty(ad)_\infty(bc)_\infty(bd)_\infty(cd)_\infty}.$$

Since

$$(-1)_\infty(\sqrt{q})_\infty(-\sqrt{q})_\infty=2(\sqrt{q};q)_\infty(-\sqrt{q};q)_\infty(-q;q)_\infty=2\quad\text{by (2.6)},$$

(2.10) is the same as the r.h.s. of (1.10). This completes the proof of (1.10).

References

1. W. A. Al-Salam and A. Verma, *Some remarks on q-beta integral*, Proc. Amer. Math. Soc. **85** (1982), 360–362.

2. G. E. Andrews and R. Askey, *Another q-extension of the beta function*, Proc. Amer. Math. Soc. **81** (1981), 97–100.

3. R. Askey, *The q-gamma and q-beta functions*, Applicable Anal. **8** (1978), 125–141.

4. ______, *An elementary evaluation of a beta type integral*, Indian J. Pure Appl. Math. **14** (1983), 892–895.

5. R. Askey and J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc. (to appear).

6. D. B. Sears, *Transformation of basic hypergeometric functions of special type*, Proc. London Math. Soc. **52** (1951),, 467–483.

7. L. J. Slater, *Generalized hypergeometric functions*, Cambridge Univ. Press, New York, 1966.

8. J. Thomae, *Beitrage zur Theorie der durch die Heinesche Reihe*: $1+((1-q^\alpha)(1-q^\beta)/(1-q)(1-q^\gamma))x+\cdots$ *darstellbaren Functionen*, J. Reine Angew. Math. **70** (1869), 258–281.

Department of Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada K1S 5B6