FINITELY ADDITIVE MEASURES ON N AND THE ADDITIVE PROPERTY

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ABSTRACT. Finitely additive measures on \( N = \{0, 1, 2, \ldots\} \) satisfying an approximation of countable additivity, called (AP), are studied. These measures are generalizations of p-points. From a p-point a translation invariant measure with (AP) is constructed. It is consistent that no measure with (AP) exists.

0. Introduction. Finitely additive measures on \( \omega = N = \{0, 1, 2, \ldots\} \) which satisfy a weak form of countable additivity are studied. A measure will always mean a finitely additive \( \mu: \mathcal{P}(\omega) \rightarrow [0, 1] \) such that \( \mu(\omega) = 1 \) and \( \mu(\{n\}) = 0 \). (Nothing is gained by letting \( \mu(\omega) \) take other values \( > 0 \).) Of course no measure is countably additive, but the following property is conceivable.

DEFINITION. A measure \( \mu \) has the additive property (denoted (AP)) if for every disjoint collection \( \{A_n: n < \omega\} \) of subsets of \( \omega \) there is \( A \subseteq \omega \) so that, for all \( n \), \( A_n \subseteq A \) (mod f.) and \( \mu(A) = \sum \mu(A_n) \). (A \( \subseteq B \) (mod f.) if \( A \setminus B \) is finite. A \( = B \) (mod f.) is defined similarly.)

Since \( B \subseteq A \) (mod f.) implies \( \mu(B) \leq \mu(A) \), (AP) is equivalent to: for every sequence \( (A_n)_{n < \omega} \) of pairwise disjoint subsets of \( \omega \) there is \( (A^*_n)_{n < \omega} \) such that, for all \( n \), \( A_n = A^*_n \) (mod f.) and \( \mu(\bigcup A^*_n) = \sum \mu(A^*_n) \). The additive property has been studied (for not necessarily total measures) in [B and F].

EXAMPLE. Any ultrafilter on \( \omega \) can be identified with a measure; i.e. sets in the ultrafilter have measure 1; other sets have measure 0. An ultrafilter with (AP) is called a p-point. p-points were introduced and studied by Walter Rudin [R] in the context of \( \beta N = N \). Since p-points have been extensively studied, there is a lot of information about some measures with (AP). Under the assumption of CH or, more generally, MA, a p-point exists (cf. [J, pp. 257–259]). However, Shelah [W, S] has shown it is consistent, assuming the consistency of ZFC, that no p-point exists.

From some points of view, ultrafilters are unsatisfactory measures. Following Maharam [M], call a measure a density if it extends natural asymptotic density; i.e. \( \mu(A) = \lim_{n \to \infty} |A \cap n|/n \) whenever this limit exists. (Here \( n = \{0, 1, \ldots, n-1\} \). A measure is translation invariant if \( \mu(A) = \mu(A + a) \) for all \( A \in \mathcal{P}(\omega) \) and \( a \in \mathbb{Z} \). (Here \( n \in A + a \) iff \( n - a \in A \).) The first section of the paper is devoted to constructing a translation invariant density from a p-point. The route taken is to construct a linear functional, show it has (AP), and then smooth it out.

The second section contains a proof that it is consistent that no measure has (AP). This proof is a generalization and simplification of Shelah's [S] for the nonex-
istence of p-points. I do not know if the existence of a measure with (AP) implies there is a p-point.

I would like to thank A. R. Freedman and John Sember for asking me about measures with (AP) and answering my many naive questions.

1. From p-points to densities. As well as measures with (AP), it is convenient to consider measures with (AP0), where (AP0) is defined just as (AP) was but the \( A_n \)'s are required to have measure 0. Consider the Banach space \( l^\infty = \{ x : \omega \to R : x \text{ is bounded in absolute value} \} \). Call a linear functional \( F : l^\infty \to R \) positive if (1) for all \( n \), \( x(n) \geq 0 \) implies \( F(x) \geq 0 \); (2) \( F(\chi_\omega) = 1 \); and (3) \( F(\chi_{\{n\}}) = 0 \) for all \( n \). As usual, \( \chi_A \) is the characteristic function of \( A \). Of course any positive linear functional \( F \) induces a measure \( \mu_F \) by \( \mu_F(A) = F(\chi_A) \). Also any measure \( \mu \) induces a positive linear functional \( F_\mu \) by \( F_\mu(x) = \int x \mu \). (Here

\[
\int x \mu = \lim_{n \to \infty} \sum_{k=-n^2}^{n^2} \frac{k}{n^2} \mu \left( x^{-1} \left( \left( \frac{k}{n^2} + 1 \right) \right) \right).
\]

It is standard and easy to see that integration theory works for \( l^\infty \) and measures (in the sense of this paper). The Riesz representation theorem explains why the positive linear functionals were singled out.

**THEOREM.** A linear functional \( F \) (on \( l^\infty \)) is positive iff there is a measure \( \mu \) such that \( F(x) = \int x \mu \) for all \( x \).

**DEFINITION.** Suppose \( F \) is a positive linear functional (on \( l^\infty \)). Temporarily call a sequence \( (x^n)_{n<\omega} \) of elements of \( l^\infty \) positive convergent if, for all \( m \) and \( n \), \( x^n(m) \geq 0 \) and the pointwise sum \( \sum x^n \in l^\infty \). Then \( F \) has (AP) if for each positive convergent sequence \( (x^n)_{n<\omega} \) there exists \( (y^n)_{n<\omega} \) so that: for all \( n \), \( y^n = x^n \) (mod f.); for all \( n \) and \( m \), \( y^n(m) \geq 0 \); and \( F(\sum y^n) = \sum F(y^n) \). One can similarly define (AP0) for positive linear functionals.

1.1. **PROPOSITION.** If a measure \( \mu \) has (AP) ((AP0)), then \( F_\mu \) has (AP) ((AP0)).

**PROOF.** Let \( F \) denote \( F_\mu \). Suppose \( (x^n)_{n<\omega} \) is as in the definition of (AP) for functionals. Choose a natural number \( N \) which is an upper bound for \( \sum x^n \). Fix \( \varepsilon > 0 \). Now a sequence \( (y^n)_{n<\omega} \) is constructed so that: \( y^n = x^n \) (mod f.); for all \( n \) and \( m \), \( y^n(m) \geq 0 \); and \( F(\sum y^n) < \sum F(y^n) + \varepsilon \). Let \( z^n = \sum_{i<n} x^i \). For all real \( r < N \), define \( A_r^n \) to be \{ \( m : z^n(m) > r \) \}. Note: for all \( n \), \( A_r^n \subseteq A_r^{n+1} \). By the (AP) of \( \mu \) we can find \( A_r \supseteq A_r^n \) (mod f.) for all \( n \) and \( \mu(A_r) = \lim \mu(A_r^n) \). Suppose \( \varepsilon > 1/N \). Let \( R = \{ i/N : 0 < i < 1/N^2 \} \). Let \( y^n(m) = 0 \) if for some \( r \in R \), \( m \in A_r^n \). Otherwise let \( y^n(m) = x^m(m) \).

**Claim.** \( \sum y^n(m) > r \) implies \( m \in A_r \) (for \( r \in R \)).

**PROOF (OF CLAIM).** First note \( \sum y^n(m) > r \) implies \( \sum x^n(m) > r \). Suppose \( m \notin A_r \). Let \( n_0 \) be the least \( n \) so that \( \sum_{i\leq n} x^i(m) > r \). So for all \( i \geq n_0 \), \( m \in A_r^i \backslash A_r \). Hence, for all \( i \geq n_0 \), \( y^i(m) = 0 \). Hence \( \sum y^n(m) \leq r \).

It can be supposed that \( A_0 = \omega \) and \( A_r \supseteq A_{r'} \) for \( r < r' \in R \). Let \( B_r = A_r \backslash A_r^{1/N} \) (where \( A_N = \emptyset \)). By the claim for \( m \in B_r \), \( \sum y^n(m) \leq r + \varepsilon \). Since \( \omega = \bigcup_{r \in R} B_r \),

\[
F \left( \sum y^n \right) \leq \sum_{r \in R} (r + \varepsilon) \mu(B_r).
\]
Also, for all \( n \),
\[
\sum_{k=0}^{n} F(y^k) = F\left(\sum_{k=0}^{n} y^k\right) \geq \sum_{r \in R} r \mu \left[ A^r_1 \setminus A^r_{1+1/2n}\right] \cap A_r,
\]
whence \( \sum F(y^n) \geq \sum_{r \in R} r \mu(B_r) \). So \( F(\sum y^n) \leq \sum F(y^n) + \varepsilon \).

To finish the proof, choose sequences \((y^n_m)_{n < \omega}\) \((1 \leq m < \omega)\) so that: \( y^n_m = x^n \) \((\text{mod } f)\) for all \( n, m \); for all \( i, n \) and \( m < m' \), \( y^n_m(i) \geq y^n_{m'}(i) \geq 0 \); and for all \( m \), \( F(\sum y^n_m) < \sum F(y^n_m) + 1/m \). So \((y^n_n)_{n < \omega}\) is the desired sequence. Exactly the same proof works for \((\text{AP})\).

Although a positive linear functional with \((\text{AP})\) has been constructed from a measure with \((\text{AP})\), it still must be “smoothed out”. Let

\[
C = \begin{bmatrix}
1 & 0 & \cdots \\
1/2 & 1/2 & 0 & \cdots \\
\vdots \\
1/n & 1/n & 1/n & 0 & \cdots \\
\vdots \\
\end{bmatrix}
\]

be the Cesàro matrix. \( C \) is a transformation on \( l^\infty \); i.e.
\[
C(x)(n) = \frac{1}{n+1} \sum_{i < n+1} x(i).
\]

1.2. LEMMA. Suppose \( F \) is a positive linear functional. Then \( F \circ C \) is a translation invariant positive linear functional such that, for all \( A \subseteq \omega \), if \( \lim |A \cap n|/n = d \) then \( F \circ C(\chi_A) = d \). Further, if \( F \) has \((\text{AP})\) \((\text{AP})\) so does \( F \circ C \).

PROOF. The first two properties result from the fact that \( F(z) = 0 \) whenever \( z \) has limit 0. It remains to see why \( F \circ C \) has \((\text{AP})\) if \( F \) does. Assume \( F \) has \((\text{AP})\). Note, given \((x^n)n < \omega\) as in the definition of \((\text{AP})\) the required \((y^n)n < \omega\) can always be chosen so that for all \( n \) there is \( N \) such that
\[
y^n(m) = \begin{cases} 
0 & \text{if } m < N, \\
x^n(m) & \text{otherwise}.
\end{cases}
\]

Temporarily call such a \( y^n, x^n \) above \( N \). Note that for all \( x \in l^\infty \) and \( N < \omega \), if \( y \) is \( x \) above \( N \) then \( F \circ C(x) = F \circ C(y) \). To see this let \( z = C(x) - C(y) \). Since \( \lim z(m) = 0, F(z) = 0 \).

Suppose now \((x^n)n < \omega\) are as in the definition of \((\text{AP})\). Let \( z^n = C(x^n) \). Choose \((N^n)n < \omega\) so that if \( w^n \) is \( z^n \) above \( N^n \) then \( F(\sum w^n) = \sum F(w^n) \). Let \( y^n \) be \( x^n \) above \( N^n \). Note that for all \( n \) and \( m \), \( C(y^n)(m) < w^n(m) \).

So
\[
F \circ C(\sum y^n) = F\left(\sum C(y^n)\right) \leq F\left(\sum w^n\right) = \sum F \circ C(y^n).
\]
Since \( \sum F \circ C(y^n) \) is always \( \leq F \circ C(\sum y^n) \), \((y^n)n < \omega\) is as required to show \( F \circ C \) satisfies \((\text{AP})\).
1.3. THEOREM. If there is a measure with \((AP)\) \((AP0)\) then there is a translation invariant density with \((AP)\) \((AP0)\).

Although I cannot show that the existence of a p-point, if some ultrafilter induces a density with \((AP0)\), then there is a p-point.

1.4. THEOREM. Suppose \(\mu\) is an ultrafilter and \(F_\mu \circ C\) has \((AP0)\). Then there is a p-point.

PROOF. Choose \(0 = \alpha_0 < \beta_0 < \alpha_1 \cdots \) so that \(n\alpha_n > \alpha_n > n\beta_{n-1}\) and \(\beta_n > n\alpha_n\). Then \((-\beta_{n-1})/i > 1 - 1/n\ (i - \alpha_n)/i > 1 - 1/n\) and \(\beta_{n-1}/i < 1/n\ (\alpha_n/i < 1/n\) for \(\alpha_n < i < \beta_n\) \((\beta_n < i < \alpha_{n+1}\)\). Either \(\bigcup_{n>0}[\alpha_n, \beta_n] \in \mu\) or \(\bigcup_{n>0}[\alpha_n, \alpha_{n+1}] \in \mu\). Assume \(X = \bigcup_{n>0}[\alpha_n, \beta_n] \in \mu\). Define \(f: \omega \to \omega\) so that

\[
\begin{cases}
  n & \text{if } \alpha_n < i < \beta_n, \\
  0 & \text{if } i \notin X.
\end{cases}
\]

Define an ultrafilter \(V\) on \(\omega\) by \(A \in V\) iff \(f^{-1}(A) \in \mu\).

Claim. \(V\) is a p-point.

PROOF (OF CLAIM). To obtain a contradiction, suppose \((A_n)_{n<\omega}\) is a collection of disjoint sets such that, for all \(n\), \(A_n \notin V\), but for any \(A \in V\) there is \(n\) so that \(A \cap A_n\) is infinite. Choose \(B_n\) so that for \(i \in [\beta_{m-1}, \beta_m)\), \(i \in B_n\) iff \(m \in A_n\). So for \(i \in [\alpha_k, \beta_k)\) and \(k \notin A_n\), \(C(x_{B_n})(i) = |B_n \cap i|/i < 1/k\). Since \(\bigcup_{k \in A_n}[\alpha_k, \beta_k] \in \mu\), \(F_\mu \circ C(x_{B_n}) = 0\). Choose \(B\) so that: \(F_\mu \circ C(x_B) = 0\); and for all \(n\), \(B \supseteq B_n\) (mod \(f\)). (Such a \(B\) exists by \((AP0)\) of \(F_\mu \circ C\).) Choose \(Y \in \mu\) so that \(Y \subseteq X\) and for all \(i \in Y\), \(|B \cap i|/i < 1/2\). By the choice of \((A_n)_{n<\omega}\) there exists an \(n\) so that \(Y \cap f^{-1}(A_n)\) is infinite. Choose \(m\) so that \(B_n \supseteq B \cup [0, \beta_{m-1})\). Pick \(k > 2\) and \(i\) so that \(i \in f^{-1}(A_n) \cap Y \cap [\alpha_k, \beta_k)\). Since \(B \supseteq [\beta_k, \beta_k)\), \(|B \cap i|/i \geq (i - \beta_{k-1})/i > 1/2\). This contradicts the choice of \(Y\).

1.5. EXAMPLE. Suppose \(\mu\) is a p-point. Choose an increasing sequence \(k(n)\) so that \(n^2/k(n) < 1/n\). Define \(\nu\) by \(A \in \nu\) iff \(\{i: \{n: k(n) + i \in A\} \in \mu\} \in \mu\). It is not hard to show \(F_\nu \circ C\) has \((AP)\).

Let \(\mu\) be any nonprincipal ultrafilter and let \((\beta_n)_{n<\omega}\) be as in Theorem 1.4. Construe \(\mu \times \mu\) (defined by \(A \in \mu \times \mu\) iff \(\{n: (n, m) \in A\} \in A\) \(\in \mu\)) as an ultrafilter \(\mu'\) on \(\omega\). Let \(\nu\) be the ultrafilter containing all sets of the form \(\{\beta_n: n \in A\}\) for some \(A \in \mu'\). Then \(F_\nu \circ C\) does not have \((AP0)\).

2. There may be no measure with \((AP)\). This section depends on some of the elements of Shelah’s proof \([S]\) that there may be no p-points. A poset \(P\) has the \(\omega^\omega\)-bounding property if, whenever \(G\) is \(P\)-generic and \(h \in V[G]\) is a function from \(\omega\) to \(\omega\), there is \(g: \omega \to \omega\) so that \(g \in V\) and \(h < g\) (i.e. for all \(n\), \(h(n) < g(n)\)). It is shown \([S, V.4]\) that an iteration with countable support of \(w\)-proper posets with the \(\omega^\omega\)-bounding property is itself \(w\)-proper and has the \(\omega^\omega\)-bounding property. (It is not necessary to know the definition of \(w\)-proper to understand this paper.)

A filter \(F\) is called a \(P\)-filter iff it is nonprincipal and for any \(\{A_n: n < \omega\}\) of elements of \(I\), the dual ideal, there is \(A \in I\) so that for all \(n\), \(A_n \subseteq A\) (mod \(f\)). Further, \(F\) is fat if, given \(\{w_n: n < \omega\}\), a set of disjoint finite subsets of \(\omega\), there is an infinite \(S \subseteq \omega\) so that \(\bigcup_{n \in S} w_n \in I\). Shelah shows if \(F\) is a fat \(P\)-filter then \(P(F)^\omega\) is an \(w\)-proper poset with the \(\omega^\omega\)-bounding property. Here
\( \mathcal{P}(F) = \{ f : f : A \to 2 \text{ for some } A \in I \} \) ordered by containment. So forcing with \( \mathcal{P}(F) \) introduces a subset of \( \omega \).

The following lemma is the analogue of VI.4.7 in [S].

2.1. \textbf{Lemma.} Suppose \( F \) is a fat \( P \)-filter and \( \mathcal{P} = \mathcal{P}(F)^\omega \times \tilde{Q} \) has the \( \omega^\omega \)-bounding property. Then \( \mathcal{P} \vDash \text{“} F \text{ cannot be extended to a measure with (AP)”} \) (i.e. there is no measure \( \mu \) with (AP) such that \( \mu(A) = 1 \) for all \( A \in F \)).

\textbf{Proof.} Assume \( G \) is \( \mathcal{P} \)-generic and \( \mu \) is a measure with (AP) extending \( F \) (in \( V[G] \)). Forcing with \( \mathcal{P}(F)^\omega \) introduces a sequence \( \langle A_0, A_1, \ldots \rangle \) of subsets of \( \omega \) defined by \( n \in A_i \) iff there is \( \langle f_0, f_1, \ldots \rangle \in G \) so that \( f_i(n) = 1 \). There are two cases. Either for all \( n \) and \( \varepsilon > 0 \) there is \( k > n \) so that \( \mu(\bigcup_{n \leq i < k} -A_i) > 1 - \varepsilon \) (\(-A_i \) denotes \( \omega \setminus A_i \)), or there is \( n \) and \( \delta > 0 \) so that, for all \( k > n \), \( \mu(\bigcap_{n \leq i < k} A_i) > \delta \). The two cases are similar, so I will only do the first one, which is the more difficult.

Assume the first case holds. Choose \( g \in V \) so that for all \( n \), \( \mu(\bigcup_{n \leq i < g(n)} -A_i) > 1 - 1/2^{n+2} \). (That such a \( g \in V[G] \) exists is implied since the first case has been assumed. The \( \omega^\omega \)-bounding property guarantees \( g \) can be chosen in \( V \).) Let \( B_n = \bigcup -A_i(g^{(n)}(0) \leq i < g^{(n+1)}(0)) \). Here \( g^{(n)} \) denotes the \( n \)th iterate of \( g \). The \( B_n \)'s have been chosen so that \( \mu(B_n) > 1 - 1/2^{n+2} \). If \( \mu \) were countably additive, then \( \mu(\bigcap B_n) \) would be \( > 1/2 \). But (AP) implies there is \( h \) (which can be taken in \( V \)) so that \( \mu(\bigcap (B_n \cup [0, h(n)])) > 1/2 \).

Let \( p = \langle f_0, f_1, \ldots, \tilde{q} \rangle \) be a condition which forces the above. Since \( F \) is a \( P \)-filter and \( \text{dom } f_i \in I \), the dual ideal, there is \( D \in I \) so that for all \( i \), \( \text{dom } f_i \subseteq D \). Choose a strictly increasing sequence \( \alpha_n \) (\( n \in \omega \)) so that: for all \( n \), \( \alpha_n > h(n) \); and for all \( i \), \( \text{dom } f_i \cap [\alpha_n, \infty) \subseteq D \). If \( g^{(n)}(0) \leq i < g^{(n+1)}(0) \), define \( f_i' = f_i \cup 1_{\alpha_n, \alpha_{n+1}} \setminus D \). Here \( 1_X \) denotes the function with domain \( X \) which is constantly 1. The choice of \( \langle \alpha_n \rangle_{n < \omega} \) guarantees \( f_i' \) is a function. So \( r = \langle f_0', f_1', \ldots \rangle \in \mathcal{P}(F)^\omega \). Note: if \( g^{(n)}(0) \leq i < g^{(n+1)}(0) \), then

\[ \tilde{A}_i \supseteq [\alpha_n, \alpha_{n+1}) \setminus D. \]

\( \langle A_i \rangle \) is a name for \( A_i \), etc.) So

\[ \vDash \bigcap (B_n \cup [0, h(n)]) \subseteq \bigcap [\alpha_n, \alpha_{n+1}) \subseteq [0, \alpha_0) \cup D \subseteq I. \]

Hence,

\[ \mu \left( \bigcap (B_n \cup [0, h(n)]) \right) = 0. \]

But \( r \) and \( p \) are consistent, a contradiction.

2.2. \textbf{Theorem.} Assume \( ZF \) is consistent. It is consistent with \( ZFC \) that no measure on \( \omega \) has (AP).

\textbf{Proof.} Beginning with \( V \models \text{GCH} \), using Lemma 2.1 and a countable support iteration of \( \omega \)-proper posets with the \( \omega^\omega \)-bounding property, one can show the following statement is consistent with \( ZFC \):

if \( F \subseteq \mathcal{P}(\omega) \) is a filter base of cardinality \( \omega_1 \) for a fat \( P \)-filter, then \( F \) cannot be extended to a measure with (AP).
So the theorem reduces to verifying the following claim.

Claim. Suppose $V \models CH$, $\mathbb{P}$ has the $\omega^\omega$-bounding property and $G$ is $\mathbb{P}$ generic. If $\mu \in V[G]$ is a measure with (AP) then there is $F \subseteq \{ A : \mu(A) = 1 \}$, a filter base of cardinality $\omega_1$ for a fat $P$-filter.

Proof (of Claim). First note: $\{ A : \mu(A) = 1 \}$ is a fat filter. If $\{ B_i : i < n+1 \}$ are disjoint subsets of $\omega$ then there is some $i$ so that $\mu(B_i) < 1/n$. Suppose $\{ w_n : n < \omega \}$ is a collection of disjoint finite subsets of $\omega$. Choose a sequence $S_1 \supseteq S_2 \supseteq \cdots$ of infinite sets so that $\mu(\bigcup_{i \in S_n} w_i) < 1/n$. Choose an infinite $S$ so that for all $n$, $S \subseteq S_n \pmod{f}$. So for all $n$,

$$\mu \left( \bigcup_{i \in S} w_i \right) \leq \mu \left( \bigcup_{i \in S \setminus S_n} w_i \right) + \mu \left( \bigcup_{i \in S_n} w_i \right) \leq 0 + \frac{1}{n}.$$ 

Hence $\mu(\bigcup_{i \in S} w_i) = 0$.

Now $F$, or more exactly a base for the dual ideal, will be defined inductively. Choose $\{ A_\alpha : \alpha < \omega_1 \}$ so that: $\mu(A_\alpha) = 0$; and if $\{ w_n : n \in \omega \} \in V$ is a collection of disjoint finite subsets of $\omega$ then there is $\alpha$ and an infinite $S \subseteq \omega$ so that $\bigcup_{n \in S} w_n \subseteq A_\alpha$. Choose countable $I_\alpha (\alpha < \omega_1) \subseteq \{ A : \mu(A) = 0 \}$ inductively so that for all $\alpha, \beta < \omega_1$, $\alpha < \beta$ implies $I_\alpha \subseteq I_\beta$; $I_\alpha$ is closed under finite unions; $A_\alpha \in I_\alpha$; and there is $A \in I_{\alpha+1}$ so that for all $B \in I_\alpha$, $A \supset B \pmod{f}$. Let $I = \bigcup_{\alpha < \omega_1} I_\alpha$ and $F$ be the dual filter base of $I$.

It is easy to see $F$ generates a $P$-filter. Suppose $\{ w_n : n \in \omega \}$ is a collection of disjoint finite subsets of $\omega$. By the $\omega^\omega$-bounding property of $\mathbb{P}$ there is $g \in V$ so that for all $i$ there is $n$ such that $[i, g(i)] \supseteq w_n$. Let $u_n = [g^{(n)}(0), g^{(n+1)}(0))$. By the choice of $I$ there is an infinite $S$ and $\alpha < \omega_1$ so that $\bigcup_{n \in S} u_n \subseteq A_\alpha$. So $S' = \{ n : w_n \subseteq A_n \}$ shows $F$ generates a fat filter.

2.3. Remark. The above proof shows that if $\mu$ is a measure, then $I = \{ A : \mu(A) = 0 \}$ is a quasi-$\sigma$-ring. That is, if $\{ A_n : n < \omega \}$ is a collection of disjoint elements of $I$, then there is an infinite $S$ so that $\bigcup_{n \in S} A_n \in I$.

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