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ON p-ADIC CONGRUENCE OF SOME CLASS FUNCTIONS ON A FINITE GROUP

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ABSTRACT. Certain p-adic integer-valued class functions on a finite group are shown to be congruent modulo a suitable power of p. This is applied to prove and extend a result of Plesken on central characters of a p-block with cyclic defect group.

G is an arbitrary finite group, p a fixed rational prime, K a p-adic number field of characteristic zero, R the ring of integers in K, π a generator of $J(R)$, and $\overline{R} = R/\pi R$. Assume that K and $\overline{R}$ are splitting fields for all subgroups of G. Let ν be the p-adic valuation on K, scaled so that $\nu(p) = 1$.

If χ is an R-valued class function on G with $\chi(1) \neq 0$ (as is the case if χ is an ordinary character of G), g is an element of G, and g^G denotes the conjugacy class of g, we define

$$\omega_\chi(g) = \chi(g)|g^G|/\chi(1).$$

Then ω_χ is a K-valued class function on G. If χ is an irreducible character, then ω_χ is R-valued, and is a central character of KG. Irreducible characters χ_1 and χ_2 are in the same p-block if and only if $\omega_{\chi_1}(g) \equiv \omega_{\chi_2}(g) \pmod{\pi R}$ for all $g \in G$ [**1**, IV.4.2].

Plesken [**2**, VIII.4] has shown (as an application of results on R-orders in the group ring RG) that if B is a p-block of G with cyclic defect group of order p^a, and if χ_u, χ_v are nonexceptional irreducible characters in B which occupy nodes in the Brauer tree for B which are not separated from each other by the exceptional node, then

$$\omega_{\chi_u}(g) \equiv \omega_{\chi_v}(g) \pmod{p^a R}, \quad \text{for all } g \in G. \tag{1}$$

Our purpose here is to show, by elementary methods of modular representation theory and well-known facts about blocks with cyclic defect groups, that (1) holds for all nonexceptional characters in B. This will be a consequence of the following general result.

THEOREM. *Let Φ be an R-linear combination of characters of projective RG-modules. Suppose that $\Phi = \varsigma + \chi$, where ς, χ are R-valued class functions on G such that $\varsigma(1)\chi(1) \neq 0$. Let $\nu(|G|) = n$ and $\nu(\chi(1)) = m$. Assume that $m < n$. If $g \in G$ is such that $\omega_\chi(g) \in R$, then $\omega_\varsigma(g) \in R$ and*

$$\omega_\varsigma(g) \equiv \omega_\chi(g) \pmod{p^{n-m} R}. \tag{2}$$

PROOF. If g is p-singular then $\Phi(g) = 0$, while if g is a p'-element then $\nu(\Phi(g)|g^G|) \geq \nu(|C_G(g)|) + \nu(|g^G|) = n$ [**1**, IV.2.5]. So in either case,

$$\chi(1)\Phi(g)|g^G| \equiv 0 \pmod{p^{n+m} R}.$$

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Thus

$$\chi(1)\varsigma(g)|g^G| \equiv -\chi(1)\chi(g)|g^G| \pmod{p^{n+m}R}. \tag{3}$$

Since $\omega_\chi(g) \in R$, we have $\nu(\chi(g)|g^G|) \geq \nu(\chi(1)) = m$, so that

$$\Phi(1)\chi(g)|g^G| \equiv 0 \pmod{p^{n+m}R}. \tag{4}$$

Then (3) and (4) yield

$$\chi(1)\varsigma(g)|g^G| \equiv (\Phi(1) - \chi(1))\chi(g)|g^G| = \varsigma(1)\chi(g)|g^G| \pmod{p^{n+m}R}. \tag{5}$$

Now divide (5) through by $\varsigma(1)\chi(1)$, note that $m < n$ and $\Phi(1) = \chi(1)+\varsigma(1)$ imply that $\nu(\varsigma(1)) = m$, and recall that $\omega_\chi(g) \in R$. It follows that $\omega_\varsigma(g) \in R$ and that (2) holds. The theorem is proved.

Now suppose that B is a p-block of G with cyclic defect group of order p^a, $a > 0$. B has nonexceptional characters χ_u, $1 \leq u \leq e$, and $(p^a - 1)/e$ exceptional characters, where $e|p-1$ is the "inertial index" of B. Also, $\nu(\chi_u(1)) = n - a$ for $1 \leq u \leq e$, where $\nu(|G|) = n$ [**1**, VII.2].

Let χ_0 equal the sum of the exceptional characters. If Φ is the character of a projective indecomposable RG-module in B, then $\Phi = \chi_u + \chi_v$ for some $0 \leq u \neq v \leq e$ [**1**, VII.2.15]. Our theorem implies that $\omega_{\chi_0}(g) \in R$ (which is also easy to see directly) and that $\omega_{\chi_u}(g) \equiv \omega_{\chi_v}(g) \pmod{p^a R}$. Connectedness of the block establishes the desired

COROLLARY. *Let χ_u, $1 \leq u \leq e$, denote the nonexceptional characters in a p-block B with cyclic defect group of order p^a, and χ_0 denote the sum of the exceptional characters in B. Then for all $g \in G$ and all $0 \leq u, v \leq e$, $\omega_{\chi_u}(g) \in R$ and $\omega_{\chi_u}(g) \equiv \omega_{\chi_v}(g) \pmod{p^a R}$.*

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