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EXTENSION OF ENTIRE FUNCTIONS ON NUCLEAR LOCALLY CONVEX SPACES

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ABSTRACT. We prove that for a nuclear locally convex complex vector space every entire function is the pull-back of some entire function on an appropriate Banach space if and only if the entire functions on E have the following universal extension property: whenever E is a topological linear subspace of a locally convex space F with a fundamental system of seminorms induced by semi-inner products, then every entire function f on E can be extended to an entire function on F.

For a complex locally convex space E let $H(E)$ denote the vector space of all entire functions on E, i.e. of all continuous complex-valued functions on E which are Gâteaux-analytic. An entire function on E is called of uniformly bounded type if it is bounded on all multiples of a suitable zero neighbourhood in E. By $H_{\mathrm{ub}}(E)$ we denote the linear space of all entire functions on E which are of uniformly bounded type.

The main result of the present article is the characterization of the nuclear locally convex spaces E satisfying the identity $H(E) = H_{\mathrm{ub}}(E)$ by the following universal extension property of the entire functions on E: Whenever E is a topological linear subspace of a locally convex space F which has a fundamental system of continuous seminorms induced by semi-inner products, then each $f \in H(E)$ has an extension to an entire function on F. This characterization is interesting as a certain contrast to a result of [**6**] by which, in every infinite-dimensional (FM)-space $F \not\simeq \mathbf{C}^{\mathbf{N}}$, there exists a closed linear subspace X such that not every entire function on X has a holomorphic extension to F.

The proof of the main result is based in one direction on an elementary lemma showing the universal extension property for every entire function of uniformly bounded type. For the other direction we use a result of Colombeau and Mujica [**4**], by which $H(F) = H_{\mathrm{ub}}(F)$ for each (DFM)-space F, together with permanence properties of the relation $H(E) = H_{\mathrm{ub}}(E)$ and the remark that every nuclear locally convex space E is isomorphic to a subspace of F^I, where F is a (DFS)-space with certain properties.

The lemma and the result of Colombeau and Mujica [**4**] mentioned above can also be used to obtain the following improvement of the extension theorem of Boland [**2**]: Let E be a (DFS)-space which has a fundamental system of continuous seminorms induced by semi-inner products and let X be a closed linear subspace of E. Then every $f \in H(X)$ can be extended to an entire function g on E. This shows

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that in Boland's theorem the nuclearity assumption can be replaced by a weaker requirement.

For a detailed investigation of the nuclear Fréchet spaces E satisfying $H(E) = H_{\mathrm{ub}}(E)$ we refer to our forthcoming article [**7**].

1. Preliminaries. We shall use the standard notation from the theory of locally convex spaces as presented in the books of Pietsch [**9**] and Schaefer [**10**]. All locally convex (l.c.) vector spaces are assumed to be complex vector spaces and Hausdorff.

(i) *Sequence spaces.* Let $A = (a_{j,k})_{(j,k)\in\mathbf{N}^2}$ be a matrix which satisfies

(1) $0 \le a_{j,k} \le a_{j,k+1}$ for all $j,\ k \in \mathbf{N}$,

(2) for each $j \in \mathbf{N}$ there exists $k \in \mathbf{N}$ with $a_{j,k} > 0$.

Then we define the sequence spaces $\lambda^s(A)$ by

$$\lambda^s(A) := \left\{ x \in \mathbf{C}^{\mathbf{N}} |\ \|x\|_k := \left(\sum_{j=1}^{\infty} (|x_j| a_{j,k})^s \right)^{1/s} < \infty \text{ for all } k \in \mathbf{N} \right\}$$

for $1 \le s < \infty$, and for $s = \infty$ by

$$\lambda^\infty(A) := \left\{ x \in \mathbf{C}^{\mathbf{N}} |\ \|x\|_k := \sup_{j\in\mathbf{N}} |x_j| a_{j,k} < \infty \text{ for all } k \in \mathbf{N} \right\}.$$

Obviously $\lambda^s(A)$ is a Fréchet space under the natural topology induced by the seminorms $(\|\ \|_k)_{k\in\mathbf{N}}$. We write $\lambda(A)$ instead of $\lambda^1(A)$.

We recall that $\lambda^s(A)$ is Schwartz, resp. nuclear, iff for every $k \in \mathbf{N}$ there exists $m \in \mathbf{N}$ and $\mu \in c_0$, resp. $\mu \in l^1$, such that $a_{j,k} \le \mu_j a_{j,m}$ for all $j \in \mathbf{N}$.

If α is an increasing unbounded sequence of positive real numbers (called an exponent sequence) and if $R = 1$ or $R = \infty$ then we define for $1 \le s \le \infty$ the power series space

$$\Lambda_R^s(\alpha) := \lambda^s(A(R,\alpha)), \quad \text{where } A(R,\alpha) = \{(r_k^{\alpha_j})_{j\in\mathbf{N}} | k \in \mathbf{N}\}, 0 < r_k \nearrow R.$$

$\Lambda_R^s(\alpha)$ is called a power series space of finite, resp. infinite, type if $R = 1$, resp. $R = \infty$.

(ii) *Holomorphic mappings.* Let E and G be l.c. spaces. A mapping $f\colon E \to G$ is called entire if f is continuous and if for every continuous linear functional y on G the function $y \circ f$ is Gâteaux-analytic. By $H(E,G)$ we denote the vector space of all entire mappings on E with values in G. Instead of $H(E,\mathbf{C})$ we write $H(E)$.

For details concerning holomorphic mappings and functions on l.c. spaces we refer to Colombeau [**3**] and Dineen [**5**].

2. Definition. (a) Let E and G be l.c. spaces. $f \in H(E,G)$ is called of uniformly bounded type if there exists an absolutely convex zero neighbourhood U in E and a bounded Banach disk B in G such that for each $r > 0$ there exists $C_r > 0$ such that $f(rU) \subset C_r B$. By $H_{\mathrm{ub}}(E,G)$ we denote the linear subspace of $H(E,G)$ consisting of all mappings of uniformly bounded type.

(b) A l.c. space is said to have property (H_{ub}) if the identity $H(E) = H_{\mathrm{ub}}(E)$ holds.

The following lemma shows that mappings of uniformly bounded type are universally extendable in a certain sense.

3. LEMMA. *Let F be a l.c. space with a fundamental system of continuous seminorms which are induced by semi-inner products and let G be a l.c. space. Then, for every topological linear subspace E of F and every $f \in H_{\mathrm{ub}}(E, G)$, there exists $g \in H_{\mathrm{ub}}(F, G)$ with $g|E = f$.*

PROOF. Let $f \in H_{\mathrm{ub}}(E, G)$ be arbitrary. Then there exist a zero-neighbourhood U in E and a Banach disk B in G such that for every $r > 0$ the set $f(rU)$ is bounded in the Banach space G_B generated by the bounded set B in G. By hypothesis we may assume that there is a continuous seminorm p on F which is induced by a semi-inner product such that U is the unit ball with respect to $p|E$. Let E_p denote the canonically normed space $E/\ker p$ and let $\pi_p^E\colon E \to E_p$ denote the canonical map. An application of Liouville's theorem shows that for every $x \in E$ and every $y \in \ker p$ we have $f(x) = f(x+y)$, since $z \mapsto f(x+zy)$ is a bounded entire function on $\mathbf{C}$. Hence there exists a function $f_p\colon E_p \to G$ with $f = f_p \circ \pi_p^E$ which is Gâteaux-analytic and for which $f_p(r\pi_p^E(U))$ is bounded in G_B for all $r > 0$. Hence $f_p = j_B \circ \tilde{f}_p$, where j_B denotes the injection $G_B \hookrightarrow G$ and where $\tilde{f}_p$ is Gâteaux-analytic and locally bounded with values in G_B. This implies $\tilde{f}_p \in H_{\mathrm{ub}}(E_p, G_B)$. From $\tilde{f}_p \in H_{\mathrm{ub}}(E_p, G_B)$ we get by the Cauchy inequalities that the polynomial Taylor series for $\tilde{f}_p$ extends to a function $\hat{f}_p \in H_{\mathrm{ub}}(\hat{E}_p, G_B)$, where $\hat{E}_p$ denotes the completion of E_p. Now we remark that $\hat{E}_p$ can be identified in a canonical and isometric way with a closed linear subspace of $\hat{F}_p$, where $F_p = F/\ker p$. $\hat{F}_p$ is a Hilbert space because of our choice of p. Hence $\hat{f}_p$ can be lifted to $\hat{g}_p \in H_{\mathrm{ub}}(\hat{F}_p, G_B)$ by means of a continuous projection on $\hat{F}_p$ with image $\hat{E}_p$. From the construction it is obvious that $g := j_B \circ \hat{g}_p \circ \pi_p^F$ belongs to $H_{\mathrm{ub}}(F, G)$ and satisfies $g|_E = f$.

In order to derive the desired characterization of the nuclear spaces with property (H_{ub}) we shall use permanence properties of (H_{ub}) and an embedding result which we are going to prove next.

4. LEMMA. *Let E be a l.c. space with property (H_{ub}). Then we have:*
(a) *The completion $\hat{E}$ of E has (H_{ub}).*
(b) *Every quotient space F of E has (H_{ub}).*
(c) *If E is stable, i.e. $E \times E \simeq E$, then E^I has (H_{ub}) for every index set I.*

PROOF. (a) This has been shown already in the proof of Lemma 3.

(b) Let $q\colon E \to F$ denote the quotient map. For $f \in H(F)$ we have $f \circ q \in H_{\mathrm{ub}}(E)$. Since q is an open map this implies $f \in H_{\mathrm{ub}}(F)$.

(c) For every $f \in H(E^I)$ an easy application of Liouville's theorem shows the existence of a finite subset J of I and of $g \in H(E^J)$ such that $f = g \circ \pi_J$, where $\pi_J\colon E^I \to E^J$ denotes the canonical projection associated with J. The stability of E implies $E^J \simeq E$ and hence $g \in H_{\mathrm{ub}}(E^J)$. Consequently f is in $H_{\mathrm{ub}}(E^I)$.

5. PROPOSITION. *Let E be a nuclear l.c. space and let α be an exponent sequence satisfying $\lim_{n\to\infty}(\log n/\alpha_n) = \infty$. Then there exists an index set I, such that E is isomorphic to a linear topological subspace of $(\Lambda^2_\infty(\alpha)'_b)^I$.*

PROOF. Since E is nuclear, we can choose an index set I and a system $(\|\ \|_i)_{i\in I}$ of seminorms generating the l.c. topology of E such that the following holds: $\hat{E}_i$, the completion of the canonically normed space $E/\ker\|\ \|_i$, is a Hilbert space and $(U_i)_{i\in I}$ is a neighbourhood basis of zero in E, where $U_i = \{x \in E \mid \|x\|_i < 1\}$. By the nuclearity of E for every $i \in I$ we can choose $j(i) \in I$ such that the canonical map $\pi_{i,j(i)}\colon \hat{E}_{j(i)} \to \hat{E}_i$ is defined and has approximation numbers in $l^{1/3}$. Applying the spectral representation theorem (see Pietsch [**9**, 8.3.1]) we get for each $i \in I$ an orthonormal system $(a_n^i)_{n\in\mathbf{N}}$ (resp. $(y_n^i)_{n\in\mathbf{N}}$) in $\hat{E}_{j(i)}$ (resp. $\hat{E}_i$) such that

$$\pi_{i,j(i)}(\xi) = \sum_{n=1}^{\infty} \lambda_n^i (\xi | a_n^i)_{j(i)} y_n^i,$$

where the sequence $(\lambda_n^i)_{n\in\mathbf{N}}$ satisfies $0 \le \lambda_n^i \le C(i)/n^3$ for some $C(i) > 0$ and all $n \in \mathbf{N}$. We claim that $((\lambda_n^i)^{1/2})_{n\in\mathbf{N}}$ is in $\Lambda_\infty^2(\alpha)$ for every $i \in I$. This holds since we have for each $r > 0$

$$\begin{aligned}\sum_{n=1}^{\infty} \left((\lambda_n^i)^{1/2} r^{\alpha_n}\right)^2 &\le \sum_{n=1}^{\infty} \frac{C(i)}{n^3} r^{2\alpha_n} \\ &= \sum_{n=1}^{\infty} \frac{C(i)}{n^2} \left(\exp\left(2\log r - \frac{\log n}{\alpha_n}\right)\right)^{\alpha_n} < \infty,\end{aligned}$$

by the properties of the exponent sequence α. Hence we can find sequences $(\mu_n^i)_{n\in\mathbf{N}}$ and $(\nu_n^i)_{n\in\mathbf{N}}$ in $\Lambda_\infty^2(\alpha)$ such that $\mu_n^i \cdot \nu_n^i = \lambda_n^i$ and $0 < \nu_n^i \le 1$ for all $n \in \mathbf{N}$ and all $i \in I$.

Now we remark that $b_n^i\colon x \mapsto \nu_n^i(\pi_{j(i)}(x) | a_n^i)_{j(i)}$ is in E' and satisfies, for all $x \in E$,

$$|b_n^i(x)| \le \nu_n^i \|\pi_{j(i)}(x)\|_{j(i)} \|a_n^i\|_{j(i)} = \nu_n^i \|x\|_{j(i)}. \tag{1}$$

From this we obtain that, for each $i \in I$, the mapping $A_i\colon x \mapsto (b_n^i(x))_{n\in\mathbf{N}}$ maps the zero neighbourhood $U_{j(i)}$ into the normal hull of the sequence $(\nu_n^i)_{n\in\mathbf{N}}$. Hence A_i is a bounded linear map of E into $\Lambda_\infty^2(\alpha)_b'$ and consequently, $A\colon E \to (\Lambda_\infty^2(\alpha)_b')^I$, $A(x) := (A_i(x))_{i\in I}$ is continuous.

Next we remark that for all $i \in I$ and all $x \in E$ we have

$$\begin{aligned}\|x\|_i^2 = \|\pi_{i,j(i)}(\pi_{j(i)}(x))\|_i^2 &= \left\|\sum_{n=1}^{\infty} \lambda_n^i (\pi_{j(i)}x | a_n^i)_{j(i)} y_n^i\right\|_i^2 \\ &= \sum_{n=1}^{\infty} \left|\mu_n^i \nu_n^i (\pi_{j(i)}(x)|a_n^i)_{j(i)}\right|^2 = \sum_{n=1}^{\infty} |\mu_n^i b_n^i(x)|^2.\end{aligned} \tag{2}$$

Since $Ax = 0$ implies $b_n^i(x) = 0$ for all $i \in I$ and all $n \in \mathbf{N}$, (2) shows that A is an injective topological homomorphism.

6. THEOREM. *For a nuclear l.c. space E the following are equivalent:*

(1) $H(E) = H_{\mathrm{ub}}(E)$.

(2) *Whenever E is a linear topological subspace of a l.c. space F with a fundamental system of continuous seminorms induced by semi-inner products, then every $f \in H(E)$ has an extension $g \in H(F)$.*

PROOF. By Lemma 3, (1) implies (2). To show the converse implication we choose I according to Proposition 5 such that E is a topological linear subspace of $(\Lambda_\infty^2(\alpha)_b')^I$ for $\alpha = ((\log n)^{1/2})_{n\in\mathbf{N}}$. Since $\sup_{n\in\mathbf{N}}(\alpha_{2n}/\alpha_n) < \infty$, the space $\Lambda_\infty^2(\alpha)_b'$ satisfies $\Lambda_\infty^2(\alpha)_b' \times \Lambda_\infty^2(\alpha)_b' \simeq \Lambda_\infty^2(\alpha)_b'$ and since $\lim_{n\to\infty} \alpha_n = \infty$, $\Lambda_\infty^2(\alpha)_b'$

is a (DFS)-space. Hence it follows from Colombeau and Mujica [**4**, 4.1] and Lemma 4(c) that $H((\Lambda^2_\infty(\alpha)'_b)^I) = H_{\text{ub}}((\Lambda^2_\infty(\alpha)'_b)^I)$. Consequently (2) implies (1).

REMARK. By Colombeau and Mujica [**4**, 4.1], every (DFM)-space has the property (H_{ub}). For nuclear Fréchet spaces, the situation is different. The space $E = H(\mathbf{C})$ does not have (H_{ub}) as an example of Nachbin [**8**] shows (see also Colombeau [**3**, 2.72] and Dineen [**5**, Example 2.22]). More generally, every nuclear space $\Lambda_\infty(\alpha)$ does not have (H_{ub}) since it is easy to check that $f_\alpha\colon z \mapsto \sum_{j=2}^\infty z_j e^{z_1\alpha_j}$ defines a function in $H(\Lambda_\infty(\alpha))$ which is not of uniformly bounded type. For examples of nuclear Fréchet spaces with property (H_{ub}) we refer to our forthcoming article [**7**] in which we study the relation $H(E) = H_{\text{ub}}(E)$ and related questions in greater detail.

An immediate consequence of Lemma 2 and Colombeau and Mujica [**4**, 4.1] is the following extension result:

7. PROPOSITION. *Let F be a l.c. space and assume that F has a fundamental system of continuous seminorms induced by semi-inner products. Let E be a linear subspace of F which is a (DFM)-space in the induced topology and let G be an arbitrary Fréchet space. Then every $f \in H(E,G)$ has an extension $g \in H(F,G)$.*

Since each closed linear subspace of a (DFS)-space is again a (DFS)-space and hence a (DFM)-space, we get from Proposition 7 the following corollary which improves the extension theorem of Boland [**2**].

8. COROLLARY. *Let F be a (DFS)-space with a fundamental system of continuous seminorms which are induced by semi-inner products, and let G be a Fréchet space. Then every $f \in H(E,G)$ has an extension $g \in H(F,G)$.*

Concluding we give some examples of nonnuclear (DFS)-spaces which satisfy the hypotheses of Corollary 8.

9. EXAMPLES. Let $A = (a_{j,k})_{(j,k)\in\mathbf{N}\times\mathbf{N}}$ be a matrix as in 1(i) with $a_{j,1} > 0$ for all $j \in \mathbf{N}$ which has the following additional properties:

(S) For every $k \in \mathbf{N}$ there exists $l \in \mathbf{N}$ with $(a_{jk}/a_{jl})_{j\in\mathbf{N}} \in c_0$.

(NN) There exists $m \in \mathbf{N}$ such that, for every $k \in \mathbf{N}$, $(a_{jm}/a_{jk})_{j\in\mathbf{N}} \notin l^1$.

Then the space $\lambda^2(A)$ is a Fréchet-Schwartz space because of (S) and it is not nuclear because of (NN). Hence the strong dual $F = (\lambda^2(A))'_b$ of $\lambda^2(A)$ is a (DFS)-space but not a (DFN)-space. From Bierstedt, Meise and Summers [**1**, 2.7], it follows that F satisfies the hypotheses of Corollary 8 since

$$F = \left\{ x \in \mathbf{C}^{\mathbf{N}}\colon\ \pi_v(x) := \left(\sum_{j=1}^\infty (|x_j|v_j)^2\right)^{1/2} < \infty \text{ for all } v \in V \right\},$$

where $V = \{v \in (0,\infty)^{\mathbf{N}}\colon \sup_{j\in\mathbf{N}} v_j a_{j,k} < \infty$ for all $k \in \mathbf{N}\}$ and where the topology of F is given by the system $(\pi_v)_{v\in V}$ of norms. Obviously every norm π_v is induced by an inner product.

Simple examples having all the properties required above are the spaces $F = (\Lambda^2_R(\alpha))'_b$, where $0 < R \le \infty$ and where the exponent sequence α satisfies

$$\sup_{n\in\mathbf{N}} \frac{\log(n+1)}{\alpha_n} = \infty.$$

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