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# TROTTER'S PRODUCT FORMULA FOR SEMIGROUPS GENERATED BY QUASILINEAR ELLIPTIC OPERATORS

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ABSTRACT. Trotter's product formula is given for nonlinear semigroups in $L^1(R^N)$ generated by quasilinear operators of the form $\Delta\phi$, where $\phi$ is a suitable function: formally $\exp(t\Delta\phi)u = \lim_{h\downarrow 0}\{\exp(h\Delta\phi_1)\cdots\exp(h\Delta\phi_k)\}^{[t/h]}u$, where $\phi = \phi_1 + \cdots + \phi_k$. The proof is carried out by a new method for construction of a semigroup with generator $\Delta\phi$ in $L^1(R^N)$.

**1. Introduction and main theorem.** Let $\phi$ be a differentiable function on $R$ with $\phi(0) = 0$ such that $\phi'$ is nonnegative and bounded on every bounded subinterval of $R$. Let $\{T(t): t > 0\}$ be the strongly continuous semigroup in the Banach space $L^1(R^N)$ with norm $\|\cdot\|_1$ defined by

$$T(t)u(x) = (4\pi t)^{-N/2}\int_{R^N} e^{-|x-y|^2/(4t)}u(y)\,dy. \tag{1.1}$$

Then the infinitesimal generator $A$ of it is the Laplacian $\Delta = \sum_{i=1}^N \partial^2/\partial x_i^2$ in $L^1(R^N)$. We consider a quasilinear operator $A_\phi$ as an operator $\Delta\phi$ in $L^1(R^N)$ defined by $A_\phi u = A\cdot\phi(u)$ for $u \in D(A_\phi)$, $D(A_\phi) = \{u \in L^1(R^N)\cap L^\infty(R^N): \phi(u) \in D(A)\}$.

We begin our theory with the generation of a nonlinear semigroup $\{S_\phi(t): t > 0\}$ in terms of $A_\phi$ from the idea of

$$h^{-1}(u(t+h,x) - u(t,x)) = L^{-1}(T(L) = I)\phi(u(t,x)),$$

which has been employed in [**3**], however, as an approximation scheme for the quasilinear parabolic equation $\partial u/\partial t = \Delta\phi(u)$. We construct $\{S_\phi(t): t > 0\}$ by means of the operator $C_{h,m}$ defined by

$$C_{h,m}u = u + hL^{-1}(T(L) - I)\phi(u) \tag{1.2}$$

with $h > 0$ and $L = h\cdot\sup_{|r|\le m}\phi'(r)$ for a positive integer $m$, and do not appeal to any result concerning the semilinear equation $\phi^{-1}(u) - \Delta u = f$ (see [**2**] with [**1**]).

The above method enables us not only to give a new proof of the generation but also to deduce some properties of the semigroup $\{S_\phi(t): t > 0\}$. As a consequence we obtain Trotter's product formula as follows.

THEOREM. *Let $\phi_j$, $j = 1, \ldots, k$, be functions on $R$ satisfying the condition for $\phi$ stated in the beginning of the present paper. Let $\{S_{\phi_j}(t): t > 0\}$, $j = 1, \ldots, k$, and $\{S_\phi(t): t > 0\}$ be the semigroups generated by $A_{\phi_j}$, $j = 1, \ldots, k$, and $A_\phi$ with $\phi = \phi_1 + \cdots + \phi_k$, respectively.*

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*Then, for every* $u \in L^1(R^N) \cap L^\infty(R^N)$,

$$\{S_{\phi_1}(h) \cdots S_{\phi_k}(h)\}^{[t/h]} u \to S_\phi(t) u \quad \textit{in } L^1(R^N) \textit{ as } h \downarrow 0$$

*uniformly on every bounded subinterval of* $[0, \infty)$.

The proof of the theorem is obtained by demonstrating that for every $\lambda > 0$ and $u \in L^1(R^N) \cap L^\infty(R^N)$,

$$(I - \lambda h^{-1}(S_{\phi_1}(h) \cdots S_{\phi_k}(h) - I))^{-1} \to (I - \lambda A_\phi)^{-1} u \tag{1.3}$$

in $L^1(R^N)$ as $h \downarrow 0$, and by then applying Brezis-Pazy's convergence theorem [**5**, Theorem 3.2]. Such a type of convergence as (1.3) has been discussed for nonlinear semigroups mainly in Hilbert spaces (see e.g. [**4** and **8**]). Recently Coron [**6**] established various product formulas in $L^1(R^N)$ for semigroups generated by quasilinear differential operators of first order.

**2. Construction of $\{S_\phi(t): t > 0\}$.** We begin this section with a lemma, of which we will make frequent use. Let $X_m$, for a positive integer $m$, be the totality of $u \in L^1(R^N) \cap L^\infty(R^N)$ such that $\|u\|_\infty \le m$, and put $X_0 = \bigcup_{m=1}^\infty X_m$. Then, $X_0$ equals $L^1(R^N) \cap L^\infty(R^N)$, a dense subspace of $L^1(R^N)$.

We consider a family $\{U(h): h > 0\}$ of operators mapping $X_m$ into itself for $m \ge 1$, and say that it satisfies the condition $(\mathrm{C})_m$ if

(i) $\|U(h)u - U(h)v\|_1 \le \|u - v\|_1$,

(ii) $\|U(h)u\|_p \le \|u\|_p$ $(p = 1, \infty)$,

(iii) $U(h)u_y = (U(h)u)_y$ for $y \in R^N$ where $u_y(x) = u(x + y)$,

(iv) $\int_{R^N} \operatorname{sgn}(u) \cdot h^{-1}(U(h) - I)u f(x)\, dx \le C_m \|u\|_1 \|\Delta f\|_\infty$ for all $h > 0$, $u, v \in X_m$ and a positive constant $C_m$, where $f$ is an arbitrary nonnegative bounded function on $R^N$ with $\Delta f \in L^\infty(R^N)$.

LEMMA 2.1. *If a family* $\{U(h): h > 0\}$ *satisfies the condition* $(\mathrm{C})_m$, *then* $J_{\lambda,h} = (I - \lambda h^{-1}(U(h) - I))^{-1}$ *is well defined for every* $\lambda > 0$, *maps* $X_m$ *into itself, and satisfies, for* $u, v \in X_m$,

(1) $\|J_{\lambda,h}u - J_{\lambda,h}v\|_1 \le \|u - v\|_1$,

(2) $\|J_{\lambda,h}u\|_p \le \|u\|_p$ $(p = 1, \infty)$,

(3) *the set* $\{J_{\lambda,h}u: h > 0\}$ *is precompact in* $L^1(R^N)$.

PROOF. For a given $u \in X_m$, $J_{\lambda,h}u$ exists as a unique fixed point of the transformation from the closed convex subset $X_m$ of $L^1(R^N)$ into itself: $v \to h(\lambda + h)^{-1}u + \lambda(\lambda + h)^{-1}U(h)v$. Clearly (1), (2) and, in particular,

$$\|(J_{\lambda,h}u)_y - J_{\lambda,h}u\|_1 \le \|u_y - u\|_1 \quad \text{for } y \in R^N \tag{2.1}$$

hold. Replacing $u$ in (iv) by $J_{\lambda,h}u$, we have

$$\int_{R^N} |J_{\lambda,h}u| f(x)\, dx \le \int_{R^N} |u| f(x)\, dx + C_m \|u\|_1 \|\Delta f\|_\infty.$$

Putting $f(x) = g(2|x|/\rho - 1)$ $(\rho > 0)$ in the above, we obtain

$$\begin{aligned} &\int_{|x|>\rho} |J_{\lambda,h}u| dx \\ &\quad \le \int_{|x|>\rho/2} |u| dx + \max\{4\rho^{-2}, 2\rho^{-1}\} \lambda C_m \|u\|_1 (\|g''\|_\infty + (N-1)\|g'\|_\infty), \end{aligned} \tag{2.2}$$

where $g$ is a function of class $C^2: R \to [0,1]$ with values 0 for $r \leq 0$ and 1 for $r \geq 1$.

Thus, by Fréchet-Kolmogorov theorem, (2.1), (2.2) and (2) ($p = 1$) imply that the set $\{J_{\lambda,h}u: h > 0\}$ is precompact in $L^1(R^N)$. Q.E.D.

To construct $\{S_\phi(t): t > 0\}$, we will deal with the operator $C_{h,m}$ defined on $X_m$ by (1.2) for each fixed $m \geq 1$.

LEMMA 2.2. *For each $m \geq 1$, the family $\{C_{h,m}: h > 0\}$ satisfies the condition* $(\mathrm{C})_m$ *with* $C_m = \sup_{|r| \leq m} \phi'(r)$.

PROOF. Since $r - hL^{-1}\phi(r)$ is nondecreasing in $r$ and hence

$$|r - s - hL^{-1}(\phi(r) - \phi(s))| + hL^{-1}|\phi(r) - \phi(s)| = |r - s| \quad \text{for } r, s \in [-m, m],$$

$C_{h,m}$ satisfies (i) and (ii). The validity of (iii) is clear from (1.1) and (1.2). It remains to show (iv). Since $h^{-1}(C_{h,m} - I) = L^{-1}(T(L) - I)\phi(\cdot)$,

$$\operatorname{sgn}(u) \cdot h^{-1}(C_{h,m} - I)u \leq L^{-1}(T(L) - I)|\phi(u)|$$

holds. Multiplication by $f(x)$ and integration of this inequality over $R^N$ gives

$$\int_{R^N} \operatorname{sgn}(u) \cdot h^{-1}(C_{h,m} - I)uf(x)\,dx \leq \int_{R^N} |\phi(u)|L^{-1}(T(L) - I)f(x)\,dx. \quad \text{Q.E.D.}$$

PROPOSITION 2.3. *$A_\phi$ is a dissipative operator with domain $D(A_\phi)$ dense in $L^1(R^N)$ satisfying the range condition*

$$R(I - \lambda A_\phi) \supset X_0 \quad \textit{for any } \lambda > 0.$$

*Moreover, for any $m \geq 1$, $(I - \lambda A_\phi)^{-1}$ maps $X_m$ into itself and satisfies, for every $u, v \in X_m$,*

$$\|(I - \lambda A_\phi)^{-1}u - (I - \lambda A_\phi)^{-1}v\|_1 \leq \|u - v\|_1,$$

$$\|(I - \lambda A_\phi)^{-1}u\|_p \leq \|u\|_p \qquad (p = 1, \infty),$$

*and*

$$(I - \lambda h^{-1}(C_{h,m} - I))^{-1}u \to (I - \lambda A_\phi)^{-1}u \quad \textit{in } L^1(R^N) \tag{2.3}$$

*as $h \downarrow 0$.*

PROOF. Let $u \in X_m$ for an arbitrary $m \geq 1$ and let $\{h_n\}_{n=1}^\infty$ be a sequence such that $h_n \downarrow 0$. Then, by Lemma 2.1, Lemma 2.2 implies that the sequence $\{J^\lambda_{h_n,m}u\}_{n=1}^\infty$, where $J^\lambda_{h,m} = (I - \lambda h^{-1}(C_{h,m} - I))^{-1}$ for $h = h_n$, contains a subsequence convergent to some $u_{\lambda,m}$ in $L^1(R^N)$. The equality

$$\begin{aligned}(I - \mu L^{-1}(T(L) - I))^{-1}\lambda^{-1}(J^\lambda_{h,m}u - u)\\ = \mu^{-1}\{(I - \mu L^{-1}(T(L) - I))^{-1} - I\}\phi(J^\lambda_{h,m}u) \quad \text{for } \mu > 0\end{aligned}$$

and the fact that, for every $v \in L^1(R^N)$,

$$(I - \mu t^{-1}(T(t) - I))^{-1}v \to (I - \mu A)^{-1}v \quad \text{in } L^1(R^N) \tag{2.4}$$

as $t \downarrow 0$ imply that

$$(I - \mu A)^{-1}\lambda^{-1}(u_{\lambda,m} - u) = \mu^{-1}((I - \mu A)^{-1} - I)\phi(u_{\lambda,m}), \tag{2.5}$$

that is, $(I-\lambda A_\phi)u_{\lambda,m}=u$ with $u_{\lambda,m}\in D(A_\phi)$ since $\|u_{\lambda,m}\|_\infty\le\|u\|_\infty$. Thus, the dissipativeness of $A_\phi$, which is obtained by letting $t\downarrow 0$ in the inequality

$$\int_{R^N}\operatorname{sgn}(v_1-v_2)\cdot t^{-1}(T(t)-I)(\phi(v_1)-\phi(v_2))\,dx\le 0$$

for $v_1,v_2\in D(A_\phi)$, implies that $u_{\lambda,m}=(I-\lambda A_\phi)^{-1}u$.

Finally, we prove that $D(A_\phi)$ is dense in $L^1(R^N)$. To this end it suffices to show that for any $u\in X_m$, $m\ge 1$, $(I-\lambda A_\phi)^{-1}u$ converges in $L^1(R^N)$ to $u$ as $\lambda\downarrow 0$. Since (2.1), (2.2) and (2) ($p=1$) remain true with $J_{\lambda,h}u$ and $C_m$ replaced by $(I-\lambda A_\phi)^{-1}u$ and $\sup_{|r|\le m}\phi'(r)$, respectively, we can prove by a similar method to that used in the proof of Lemma 2.1 that the set $\{(I-\lambda A_\phi)^{-1}u:\lambda>0\}$ is precompact in $L^1(R^N)$. Therefore for any sequence $\{\lambda_n\}_{n=1}^\infty$ such that $\lambda_n\downarrow 0$, the sequence $\{(I-\lambda_nA_\phi)^{-1}u\}_{n=1}^\infty$ contains a subsequence convergent to some $w$ in $L^1(R^N)$. The equality (2.5) with $u_{\lambda,m}$ replaced with $(I-\lambda A_\phi)^{-1}u$ implies that $(I-\mu A)^{-1}(w-u)=0$ for $\mu>0$. Q.E.D.

**3. Properties of $\{S_\phi(t): t>0\}$.** In the preceding section we have proved that $A_\phi$ generates a semigroup $\{S_\phi(t): t>0\}$ in the sense of Crandall-Liggett [**7**, Theorem I]. The purpose of this section is to study further properties of $\{S_\phi(t): t>0\}$ and to give the proof of Theorem. With the aid of the Brezis-Pazy's convergence theorem, the following can be obtained from Proposition 2.3.

PROPOSITION 3.1. *For every $t>0$, $S_\phi(t)$ maps $X_m$ into itself for any $m\ge 1$ and satisfies, for every $u,v\in X_m$,*

$$\|S_\phi(t)u-S_\phi(t)v\|_1\le\|u-v\|_1,\quad \|S_\phi(t)u\|_p\le\|u\|_p\qquad (p=1,\infty)$$

*and*

$$C_{h,m}^{[t/h]}u\to S_\phi(t)u\quad\text{in } L^1(R^N)\tag{3.1}$$

*as $h\downarrow 0$ uniformly on every bounded subinterval of $[0,\infty)$.*

LEMMA 3.2. *For every $m\ge 1$, $\{S_\phi(t): t>0\}$ satisfies the condition* (C)$_m$ *with $C_m=\sup_{|r|\le m}\phi'(r)$.*

PROOF. In view of (3.1) we see that $S_\phi(t)u_y=(S_\phi(t)u)_y$ for $y\in R^N$ since $C_{h,m}u_y=(C_{h,m}u)_y$. It remains to show that, for $u\in X_m$,

$$\int_{R^N}\operatorname{sgn}(u)\cdot t^{-1}(S_\phi(t)-I)uf(x)\,dx\le\sup_{|r|\le m}\phi'(r)\cdot\|u\|_1\,\|\Delta f\|_\infty.\tag{3.2}$$

Since $C_{h,m}^n-I=\sum_{k=0}^{n-1}(C_{h,m}-I)C_{h,m}^k$ for any positive integer $n$, we have, by Lemma 2.2,

$$\int_{R^N}\operatorname{sgn}(u)\cdot(C_{h,m}^n-I)uf(x)\,dx\le nh\sup_{|r|\le m}\phi'(r)\cdot\|u\|_1\,\|\Delta f\|_\infty.$$

Putting $n=[t/h]$ and letting $h\downarrow 0$ yields (3.2). Q.E.D.

LEMMA 3.3. *For every $u\in X_0$, $\int_0^t\phi(S_\phi(r)u)\,dr$ belongs to $D(A)$, and for all $t\ge 0$,*

$$S_\phi(t)u-u=A\int_0^t\phi(S_\phi(r)u)\,dr\quad\text{in } L^1(R^N).\tag{3.3}$$

PROOF. Let $u \in X_m$ for an arbitrary $m \geq 1$ and let $u_h$ be, for $h > 0$, the solution in $C([0,\infty); X_m)$ of

$$u(t) = e^{-t/h}u + h^{-1}\int_0^t e^{-(t-r)/h}C_{h,m}u(r)\,dr, \qquad t \geq 0.$$

It is easy to verify that $u_h(t)$ is differentiable in $t$ in the topology of $L^1(R^N)$ and satisfies

$$u_h(t) - u = L^{-1}(T(L) - I)\int_0^t \phi(u_h(r))\,dr, \qquad t \geq 0. \tag{3.4}$$

By a well-known convergence theorem (see e.g. [**5**, Theorem 3.1]), (2.3) implies that

$$u_h(t) \to S_\phi(t)u \quad \text{in } L^1(R^N) \tag{3.5}$$

as $h \downarrow 0$ uniformly on every bounded subinterval of $[0,\infty)$. From (3.4) it follows that for any $\mu > 0$,

$$\begin{aligned}(I - \mu L^{-1}&(T(L) - I))^{-1}(u_h(t) - u)\\ &= \mu^{-1}\{(I - \mu L^{-1}(T(L) - I))^{-1} - I\}\int_0^t \phi(u_h(r))\,dr,\end{aligned}$$

which together with (2.4) and (3.5) implies (3.3). Q.E.D.

LEMMA 3.4. *Under the assumptions of Theorem, the family* $\{S_{\phi_1}(t)\cdots S_{\phi_k}(t)\colon t > 0\}$ *satisfies the condition* $(\mathrm{C})_m$ *with* $C_m = \sum_{j=1}^k \sup_{|r|\leq m}\phi_j'(r)$ *for every* $m \geq 1$.

PROOF. The validity of (i)–(iii) of $(\mathrm{C})_m$ is clear from Lemma 3.2. Since

$$S_{\phi_1}(t)\cdots S_{\phi_k}(t) - I = \sum_{j=1}^k (S_{\phi_j}(t) - I)S_{\phi_{j+1}}(t)\cdots S_{\phi_k}(t),$$

$k$ applications of (3.2) yields

$$\begin{aligned}\int_{R^N} &\operatorname{sgn}(u)\cdot(S_{\phi_1}(t)\cdots S_{\phi_k}(t) - I)uf(x)\,dx\\ &\leq t\sum_{j=1}^k \sup_{|r|\leq m}\phi_j'(r)\cdot\|u\|_1\,\|\Delta f\|_\infty.\end{aligned}$$

Q.E.D.

PROOF OF THEOREM. As was mentioned in the first section, we have only to show (1.3). Let $u \in X_0$. Then, $u \in X_m$ for some $m \geq 1$. By Lemma 3.4, for any sequence $\{h_n\}_{n=1}^\infty$ such that $h_n \downarrow 0$, the sequence $\{J_{h_n}^\lambda u\}_{n=1}^\infty$, where

$$J_h^\lambda = (I - \lambda h^{-1}(S_{\phi_1}(h)\cdots S_{\phi_k}(h) - I))^{-1}$$

for $h = h_n$, contains a subsequence convergent to some $u_\lambda$ in $L^1(R^N)$. By Lemma 3.3 it holds that, for $h > 0$,

$$\begin{aligned}\lambda^{-1}(J_h^\lambda u - u) &= h^{-1}(S_{\phi_1}(h)\cdots S_{\phi_k}(h) - I)J_h^\lambda u\\ &= h^{-1}\sum_{j=1}^k (S_{\phi_j}(h) - I)S_{\phi_{j+1}}(h)\cdots S_{\phi_k}(h)J_h^\lambda u\\ &= A\cdot\sum_{j=1}^k h^{-1}\int_0^h \phi_j(S_{\phi_j}(r)S_{\phi_{j+1}}(h)\cdots S_{\phi_k}(h)J_h^\lambda u)\,dr.\end{aligned} \tag{3.6}$$

Since $A$ is closed and $\|u_\lambda\|_\infty \le \|u\|_\infty$, (3.6) implies that

$$\lambda^{-1}(u_\lambda - u) = A \cdot \sum_{j=1}^{k} \phi_j(u_\lambda) = A_\phi u_\lambda,$$

and hence $u_\lambda = (I - \lambda A_\phi)^{-1} u$. Q.E.D.

Quite similarly, with the aid of Lemma 2.1 we can also obtain the following formula under the assumptions of Theorem:

$$\{k^{-1}(S_{\phi_1}(kh) + \cdots + S_{\phi_k}(kh))\}^{[t/h]} u \to S_\phi(t)u \quad \text{in } L^1(R^N)$$

as $h \downarrow 0$ for $u \in L^1(R^N) \cap L^\infty(R^N)$, uniformly on every bounded subinterval of $[0, \infty)$. In fact, the family $\{k^{-1}(S_{\phi_1}(kt) + \cdots + S_{\phi_k}(kt)) : t > 0\}$ satisfies the condition $(\mathrm{C})_m$ with $C_m = \sum_{j=1}^{k} \sup_{|r| \le m} \phi_j'(r)$.

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